The Role of Topology in Network Science

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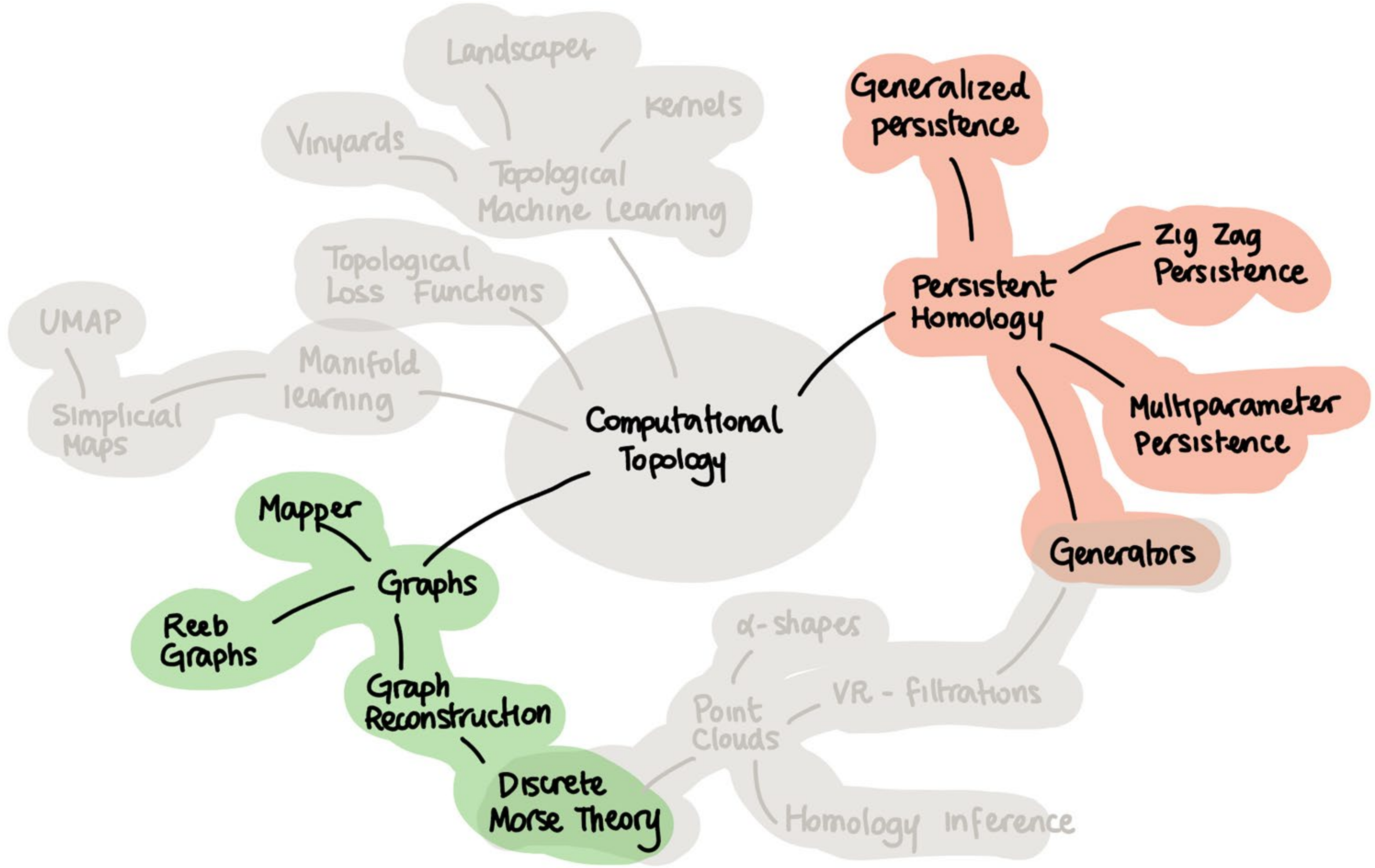
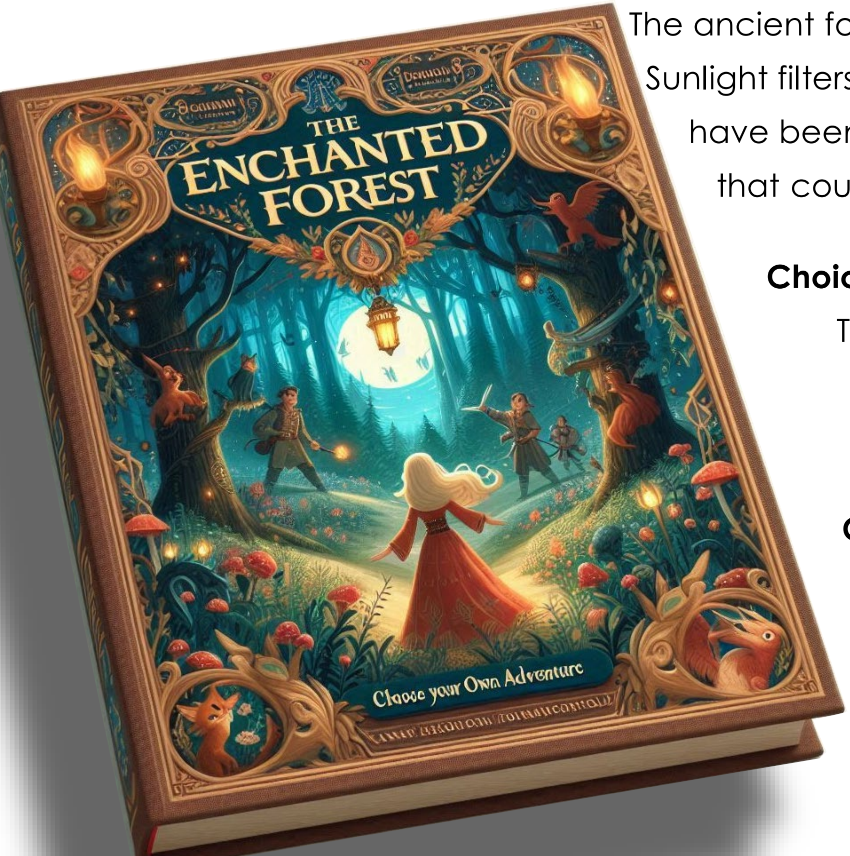

Landscapes
Kernels
Vinyards
Topological Machine Learning
Topological Loss Functions
UMAP
Simplicial Maps
Manifold learning
Computational Topology
Generalized persistence
Zig Zag Persistence
Persistent Homology
Multiparameter Persistence
Generators
Mapper
Graphs
Reeb Graphs
Graph Reconstruction
Discrete Morse Theory
α-shapes
VR - filtrations
Point Clouds
Homology inference

The Forking Path



The ancient forest looms before you, its gnarled trees stretching toward the sky like ancient guardians. Sunlight filters through the thick canopy, casting dappled shadows on the moss-covered ground. You have been following the faint trail for hours, and now you stand at a crossroads—a fork in the path that could lead you deeper into mystery or back to safety.

Choice 1: The Whispering Grove

To your left, the path winds through a grove of silver-barked trees. You hear faint whispers carried on the breeze. Do you follow this path, drawn by curiosity and the promise of hidden knowledge? **(continue to <u>Geometric Realizations</u>)**

Choice 2: The Forgotten Bridge

Straight ahead, a rickety wooden bridge spans a rushing stream. The water below is crystal clear, bringing forth memories of a forgotten past. Will you risk the bridge's creaking planks and venture toward fortune? **(continue to <u>Topology 101</u>)**

Choice 3: You turn around and run home (turn to your phone or laptop)

Choose wisely, dear reader. Your fate awaits.

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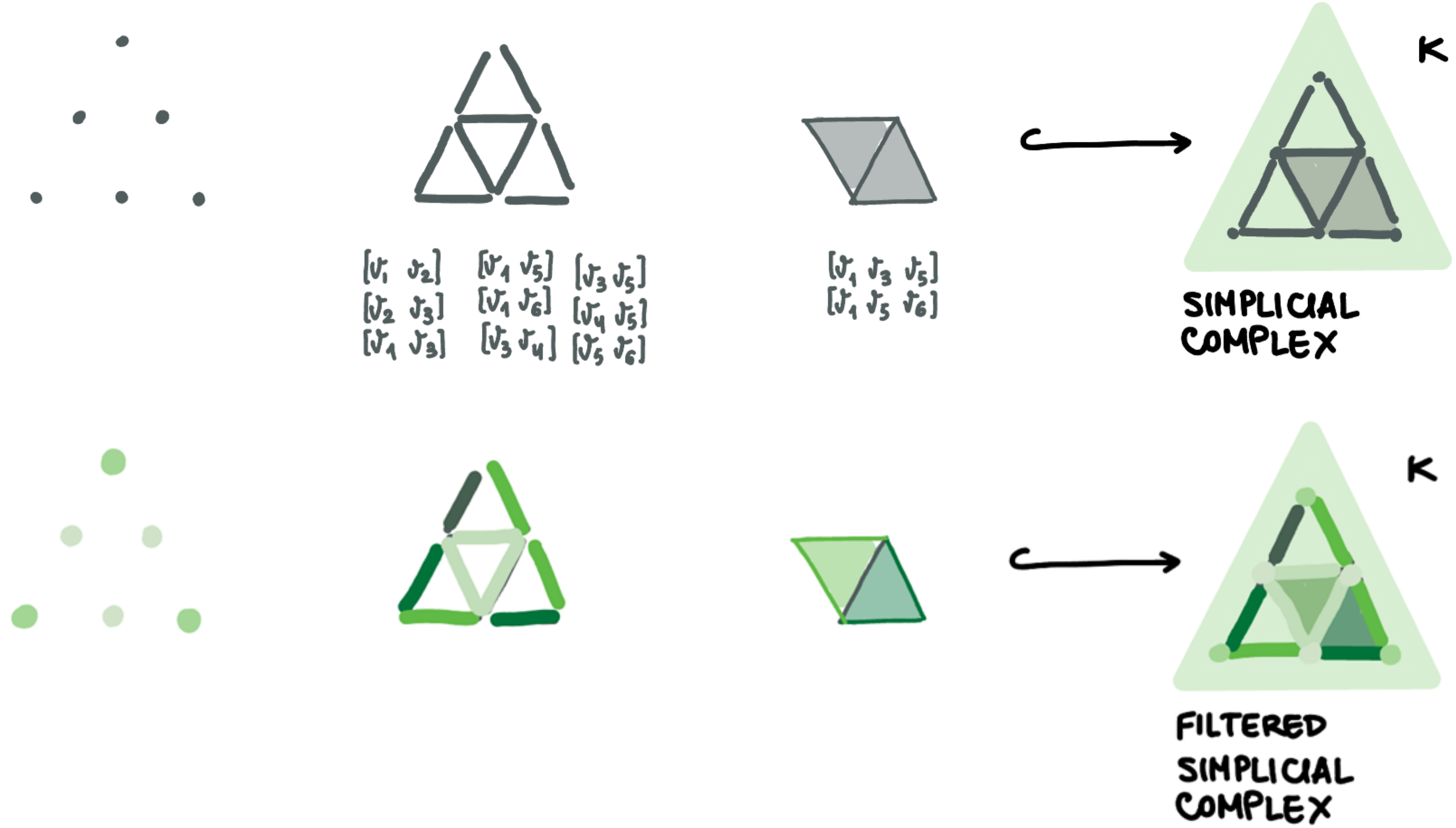
At the beginning there were Higher-order relations



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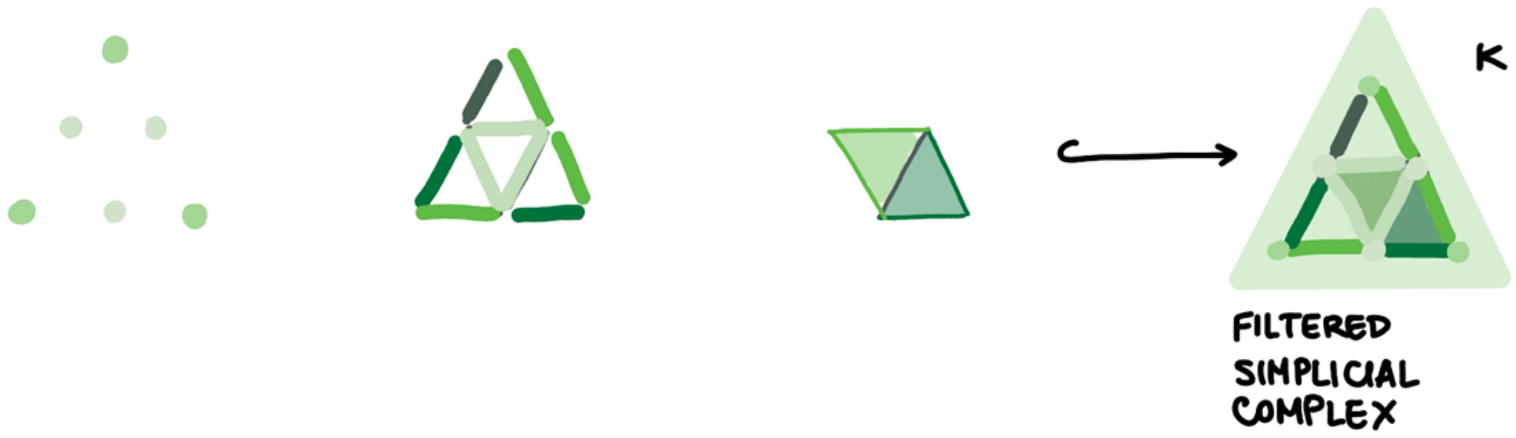
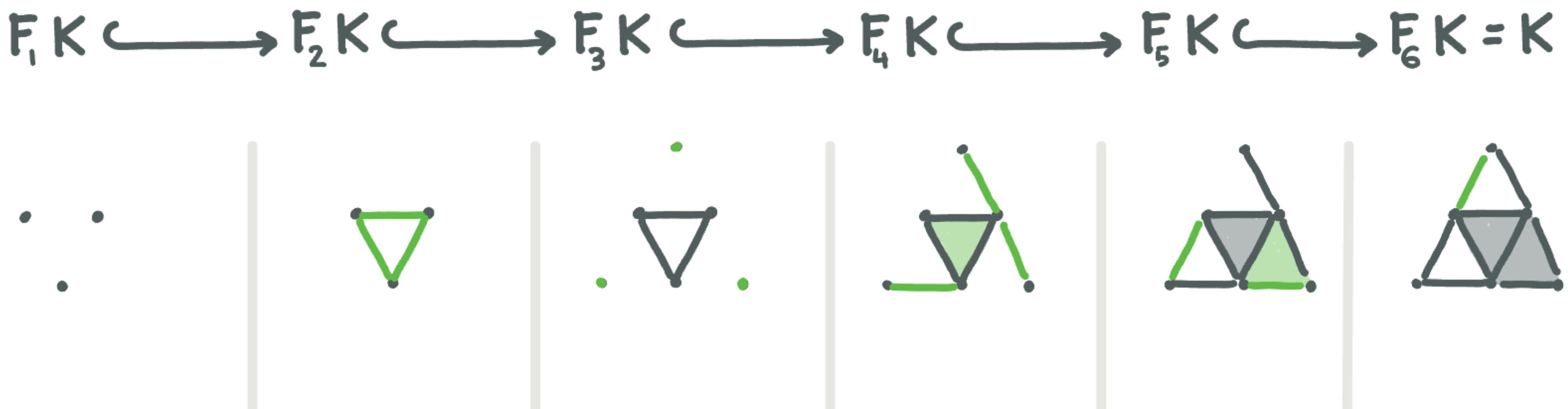
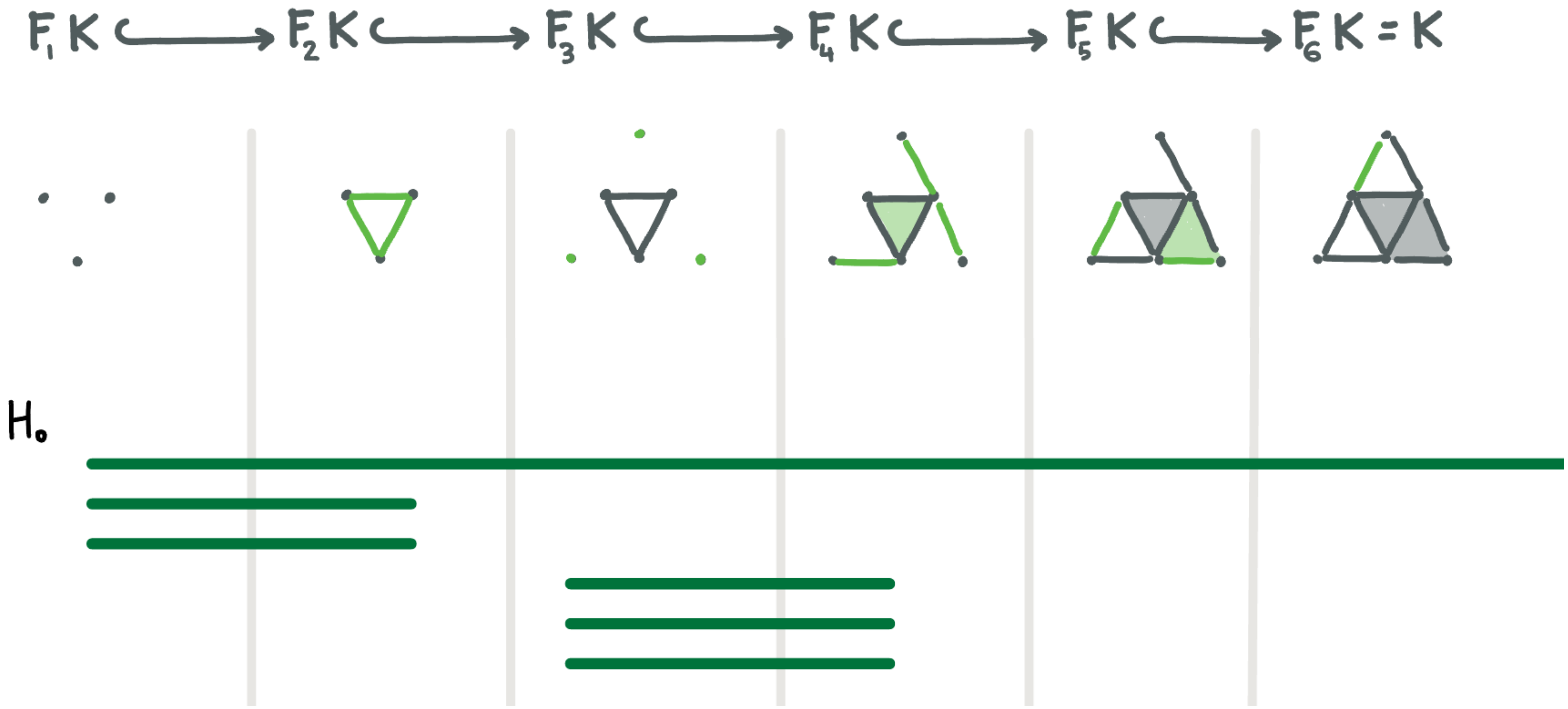
Persistent Homology

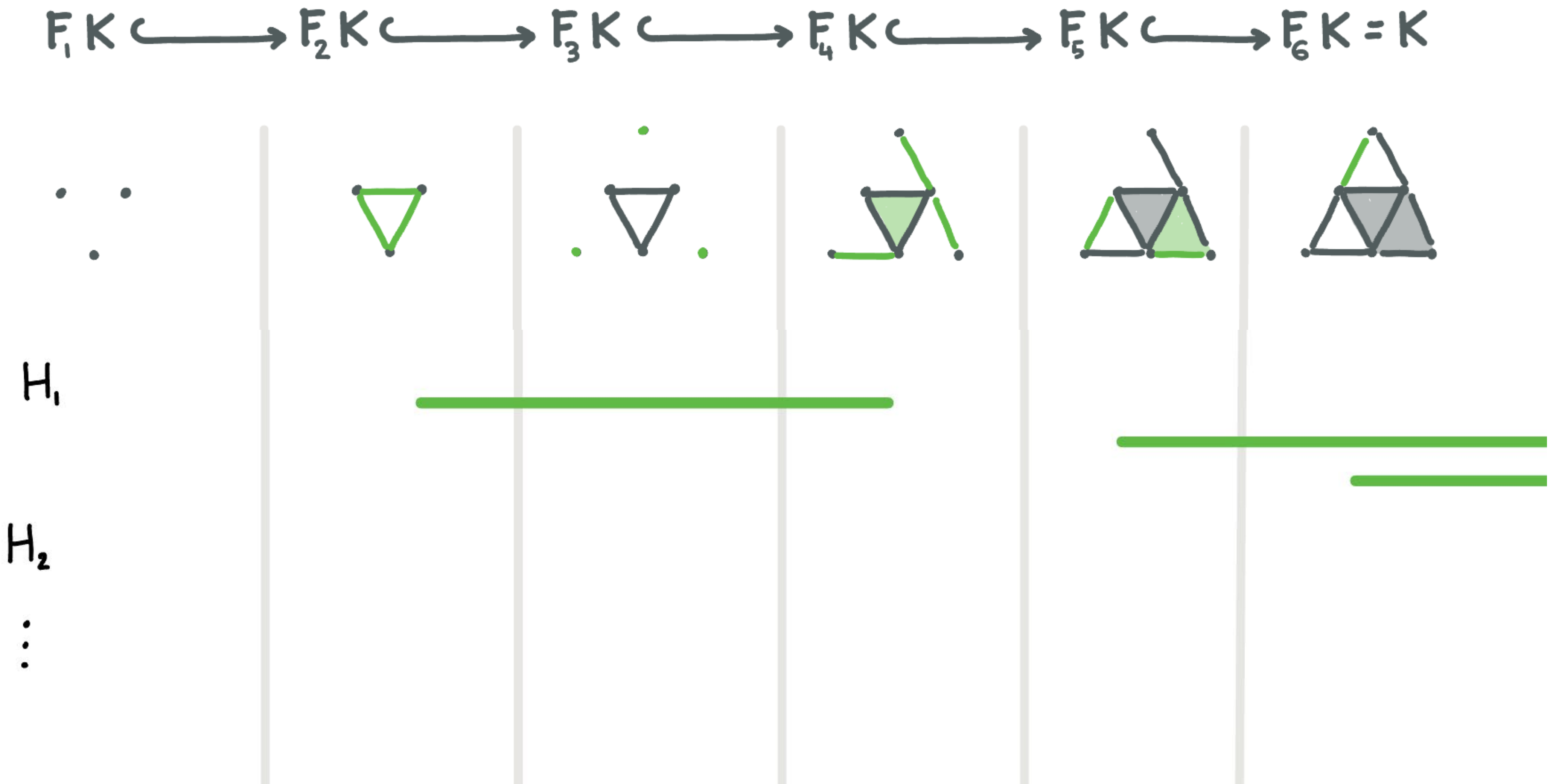




Persistent Homology



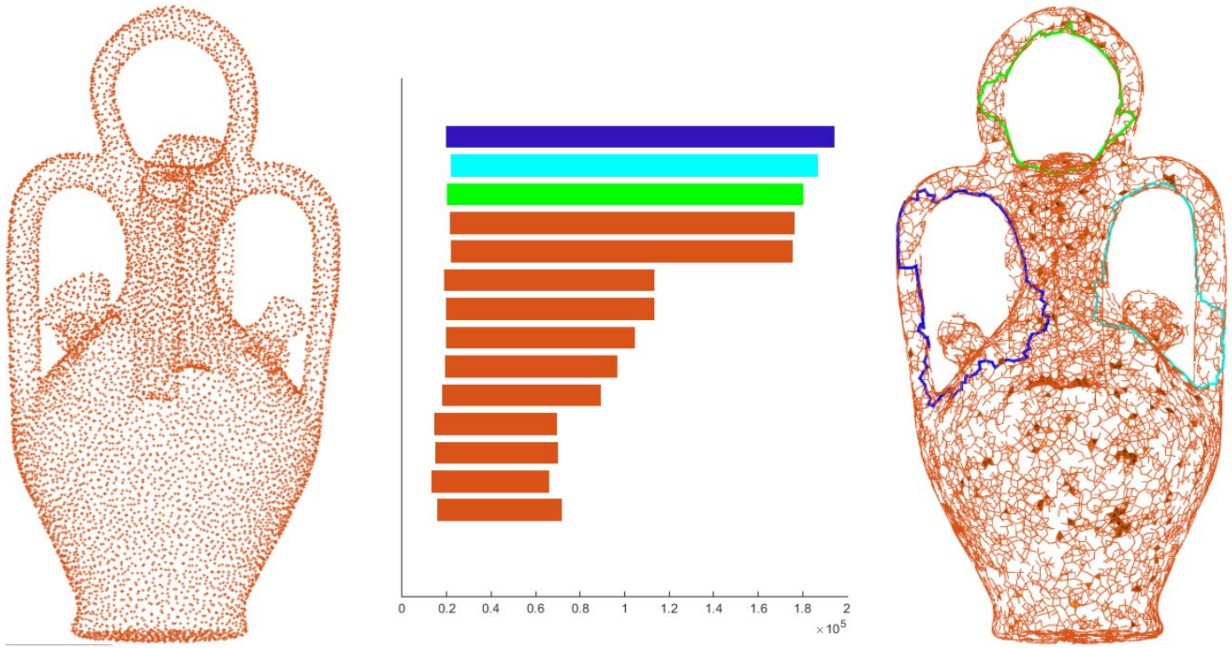
Persistent Homology



Bringing Geometry back into Topology

Going beyond counting homological cycles

For the first decade of its life, TDA focused mostly on finding ways to count cycles and studying the stability under noise. In the last decade, focus has shifted to identifying a **optimal homology basis** [Eriksson Whittlesey 2005] and **minimal cycles** [Dey et al. 2018] [Guerra et al. 2021] [Li et al. 2021].



(a) (b) (c)

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Bringing Geometry back into Topology

Duality between persistent cycles and flow network cuts

Both these problems are NP-hard in general, but for **weak p-pseudomanifolds** * optimal **p-1** homological cycles can be computed in polynomial time. [Chen and Freedman 2011]

A **flow network** $(\mathrm{G}, \mathrm{s}, \mathrm{t}, \mathrm{C})$ is an undirected graph G , two subsets of the vertices $\mathrm{s}, \mathrm{t} \subset \mathrm{V}(\mathrm{G})$ called *sources* and *sinks* respectively, and a *capacity* function $\mathrm{C}: \mathrm{E}(\mathrm{G}) \rightarrow [0, +\infty)$.

A **cut** of G is a partition of the vertices $\mathrm{S}, \mathrm{T} \subset V(G)$ so that $\mathrm{s} \subset \mathrm{S}$, $\mathrm{t} \subset \mathrm{T}$, and $\mathrm{S} \cup \mathrm{T} = \mathrm{V}(\mathrm{G})$.

The **capacity** of a cut (S, T) is the sum of the capacity of the edges between (S, T)

$$C(S,T) = \sum_{e \in E(S,T)} C(e).$$

A **minimal cut** of $(\mathrm{G}, \mathrm{s}, \mathrm{t}, \mathrm{C})$ is a cut (S, T) with minimal capacity.

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* *Network Geometry* [Bianconi Rahmede 2016]

Bringing Geometry back into Topology

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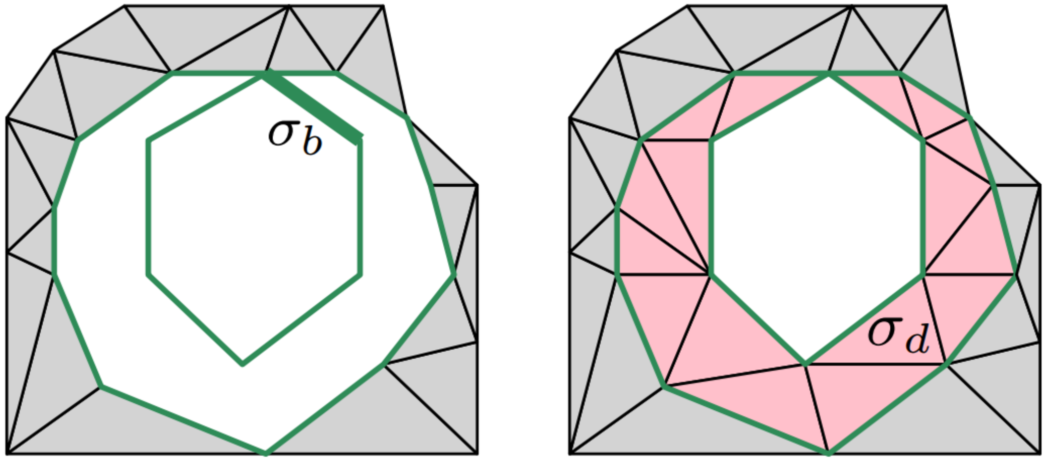

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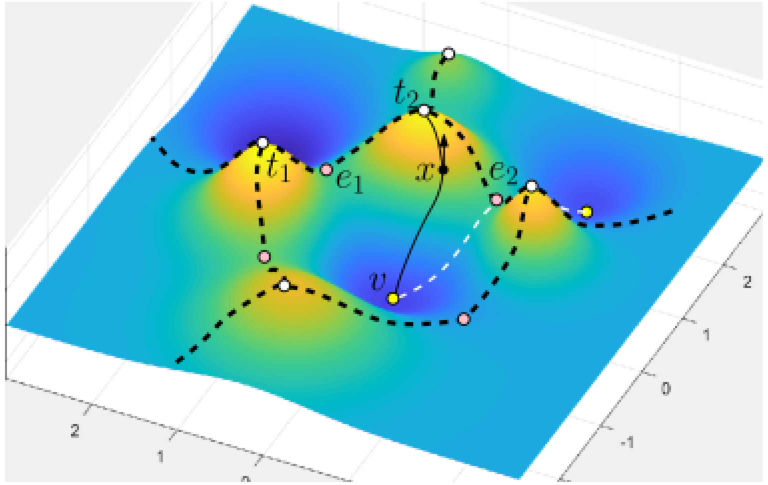
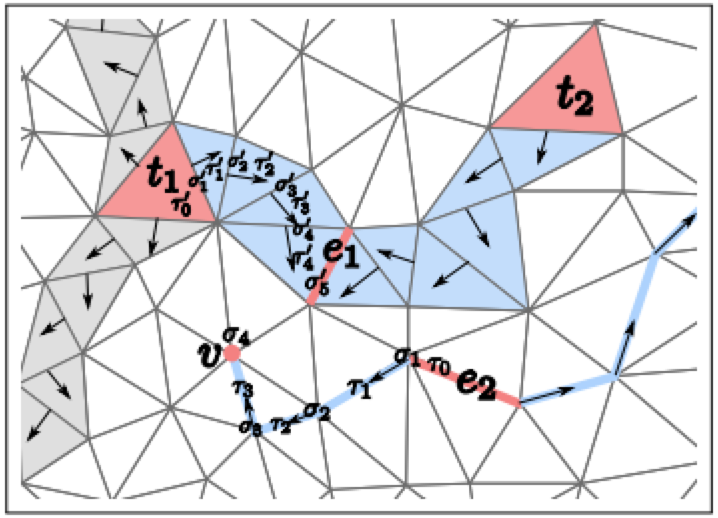
Computational Topology for Data Analysis §5.3 [Dey Wang 2022]

The importance of a manifold

This intuition has been expanded to applications of Discrete Morse Theory to TDA.

Morse Theory is the branch of mathematics that connects the topology of smooth manifolds to the behavior of certain functions defined on those manifolds.





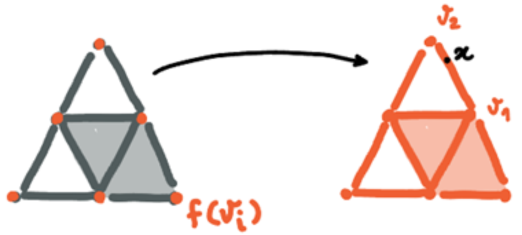
Computational Topology for Data Analysis §10 [Dey Wang 2022]

The importance of a manifold

Piecewise-Linear functions

A **PL-function** $\bar{f}: K \to \mathbb{R}$ is a real-valued function on the geometric realization of a simplicial complex determined by the its restriction $f: V \to \mathbb{R}$ to the vertex set V and linearly extending it within each simplex in K.

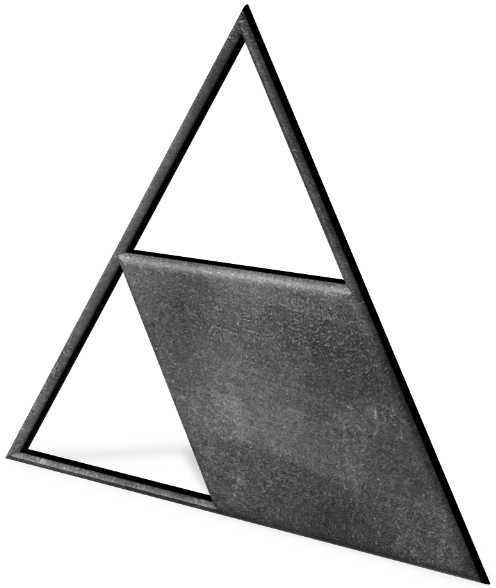
We have weights on the vertices



We extend them linearly to all K

$$\bar{f}(x) = \sum_{1}^{K+1} \alpha_i f(v_i) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$$

barycentric coordinate



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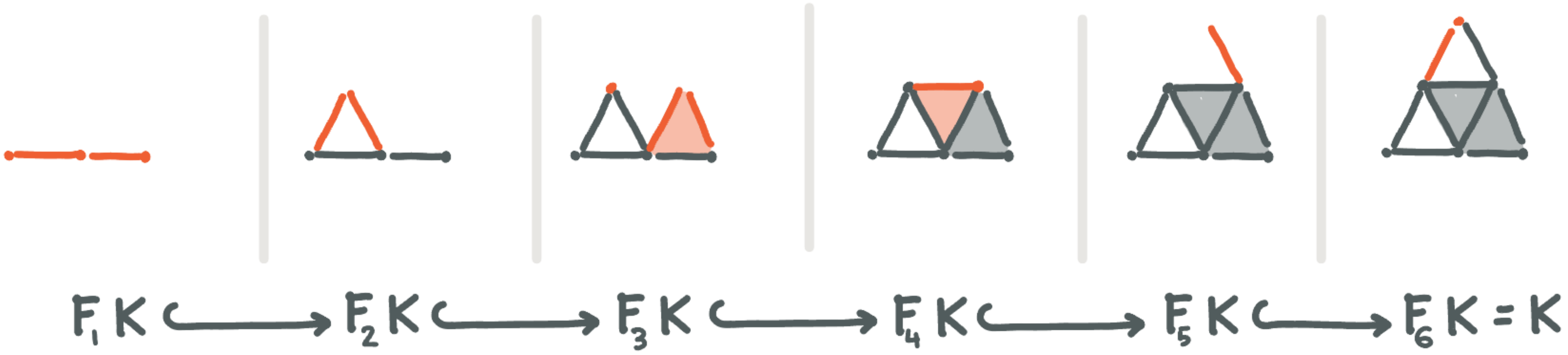
Piecewise-Linear functions

A **PL-function** $\bar{f}: K \to \mathbb{R}$ defines a filtration on the geometric realization of K



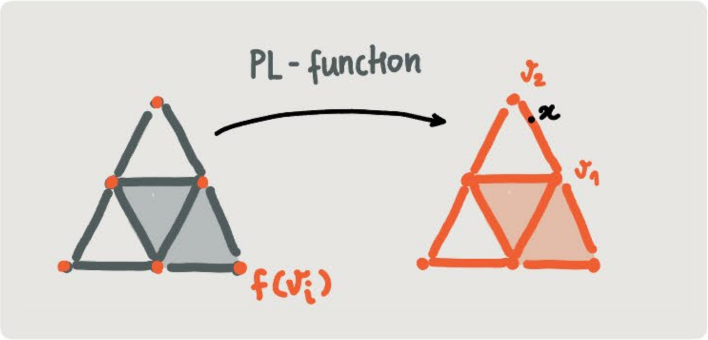
Closed star

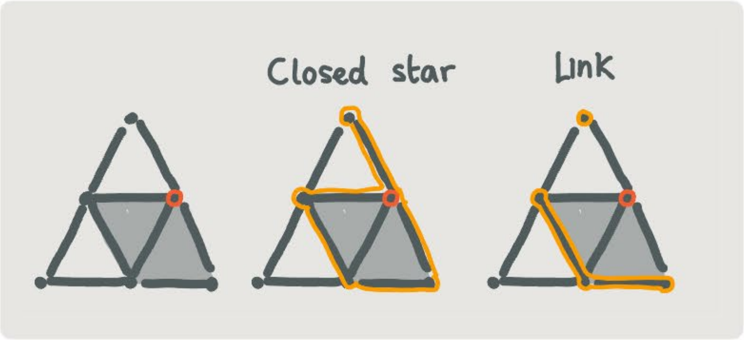
the simplicial complex induced by all the simplices that contain σ

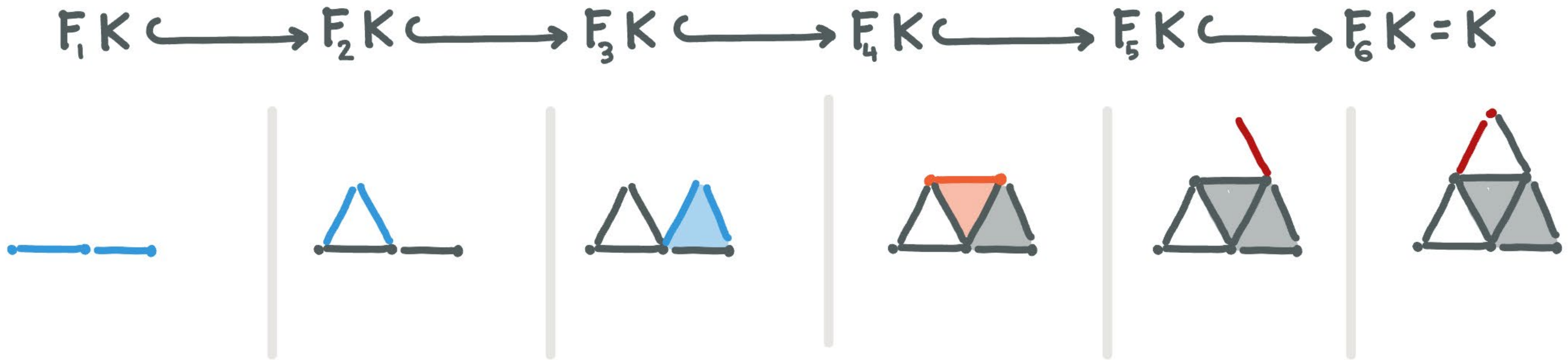
Link

the simplices in the closed star that don't intersect σ

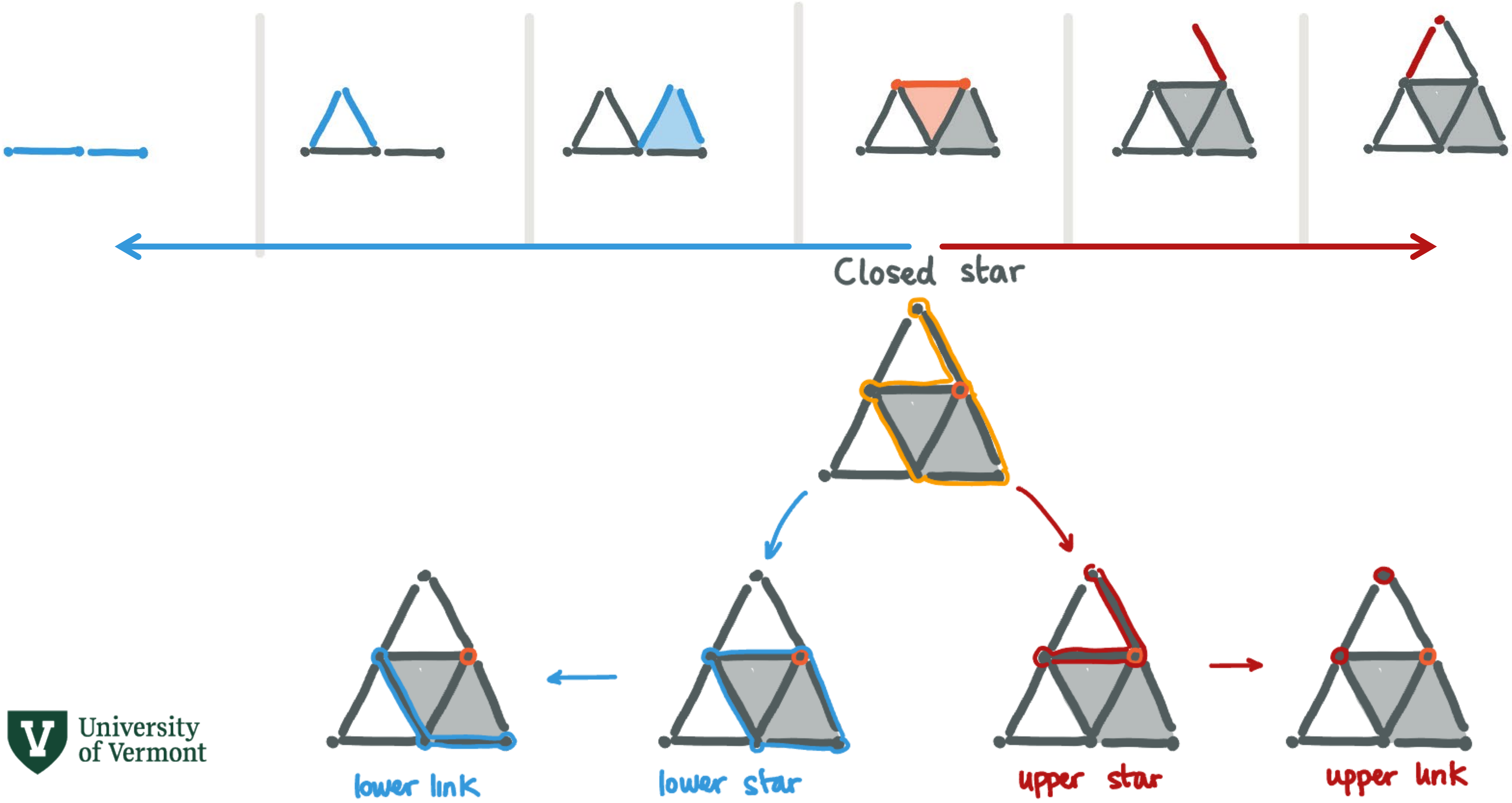
Piecewise-Linear functions







Piecewise-Linear functions



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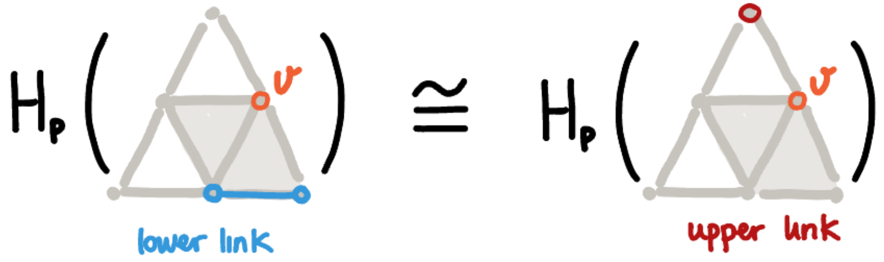
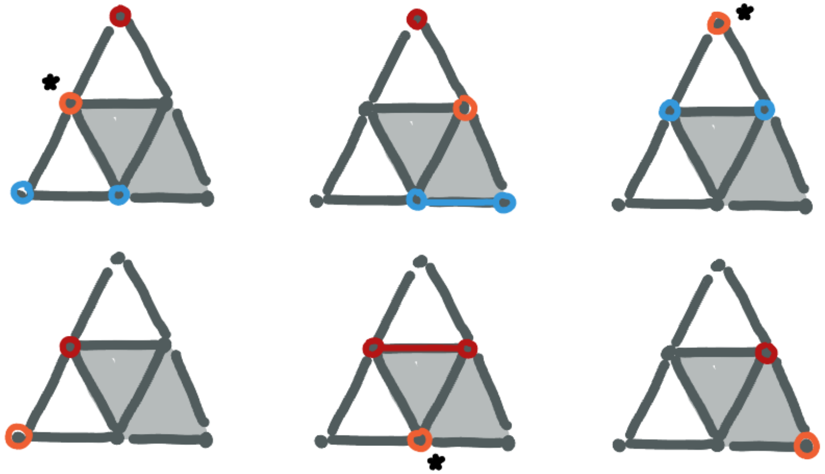
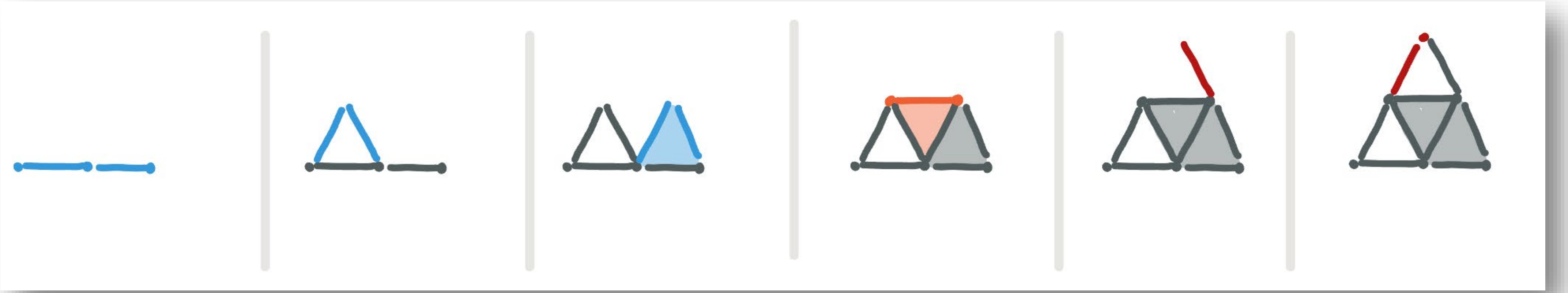
Critical Points



for all $p>0$ then v is a regular point of K

otherwise v is a critical point of K





Why Should I care?

PL-functions and Networks

Computing 0th persistent homology for a PL-function we retrieve the Kruskal's **Minimum Spanning Forest algorithm** for the graph underlying K.
This means that computing the persistent homology of a graph can be computed in O(nlog(n)). [Dey and Wang, 2022]

Generalizations of Persistent Homology

There are 2 ways to generalize a filtration: the functions [Dey, Fan, Wang 2012] or the directions. [Carlsson and De Silva, 2010]*

Graph Reconstruction

Suppose we have a hidden geometric graph G embedded in $\mathbb{R}^n$ and a density PL-function ρ that concentrates around the hidden graph G. We can extend the Kruskal algorithm to reconstruct G from ρ. [Wang et al., 2015]**

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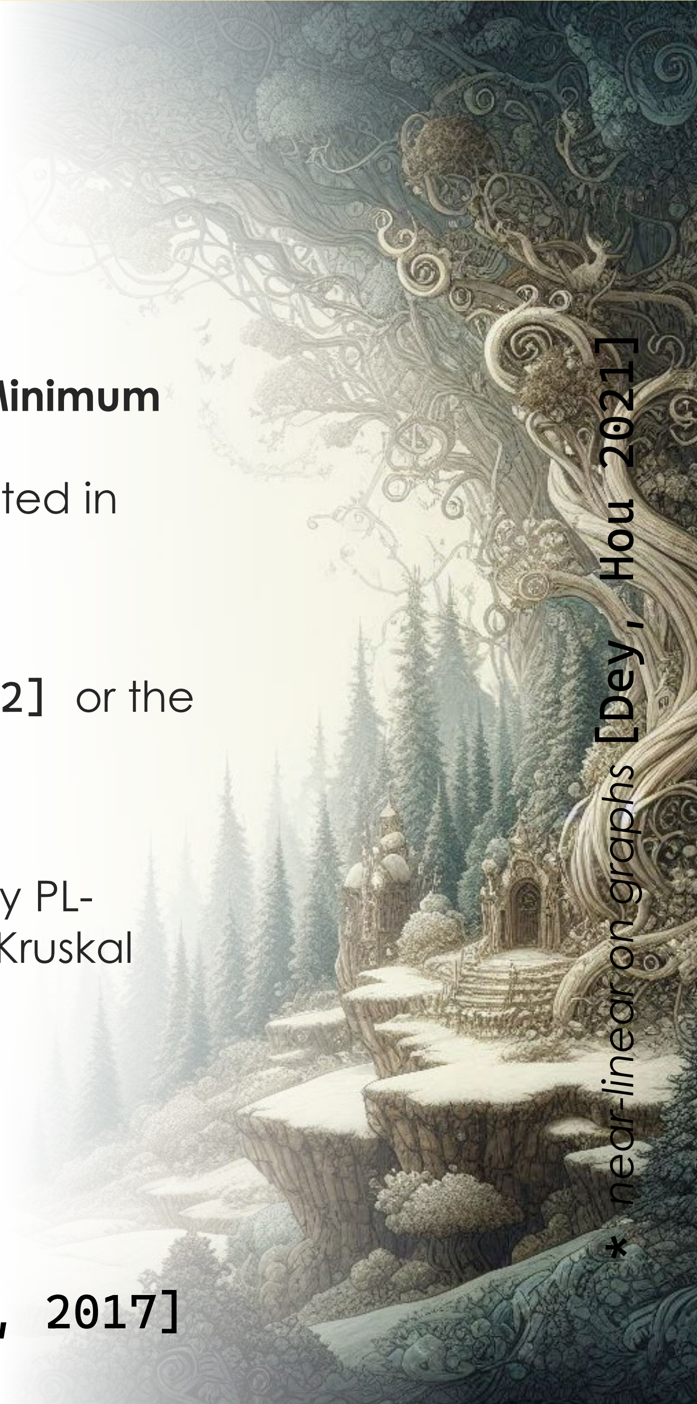
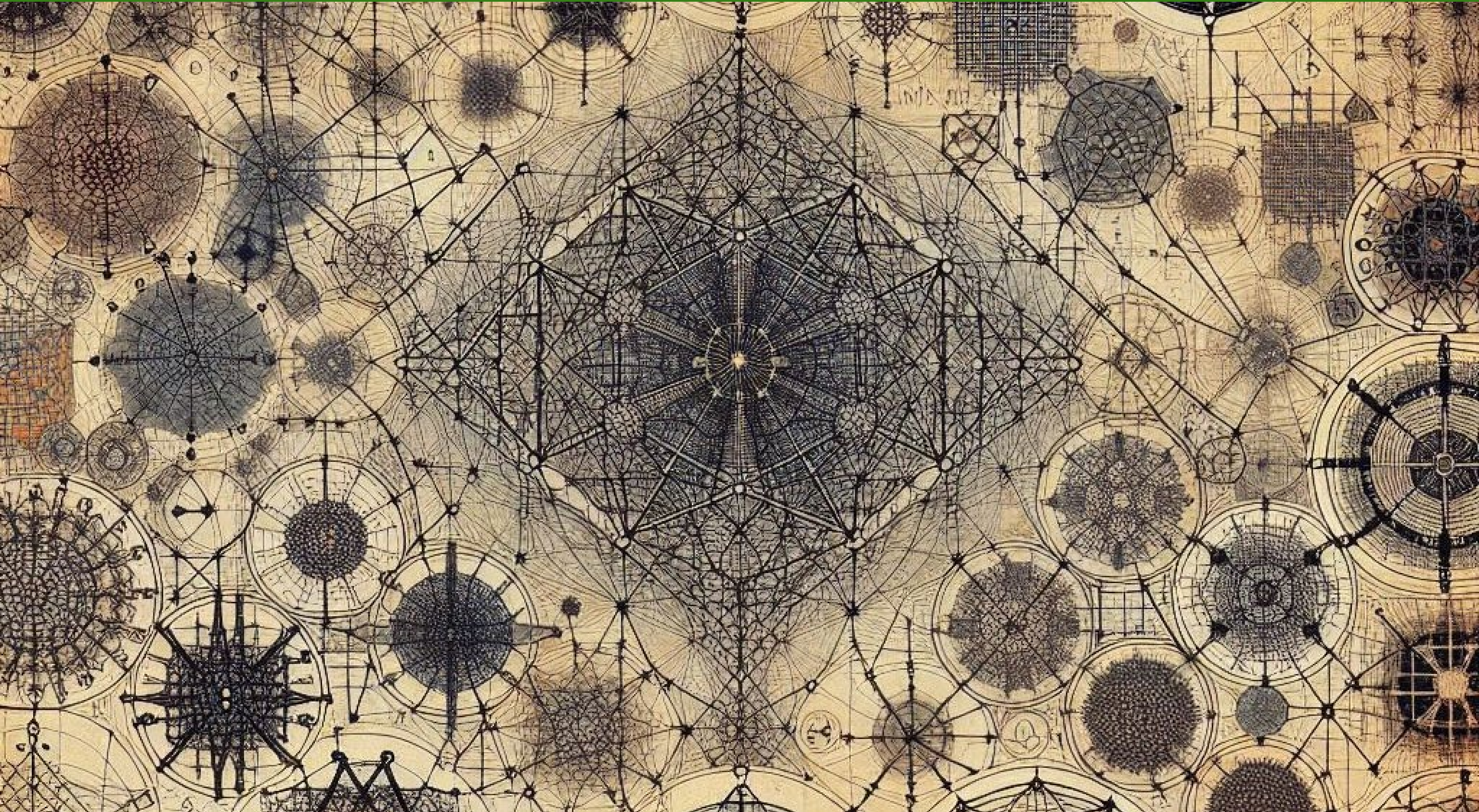
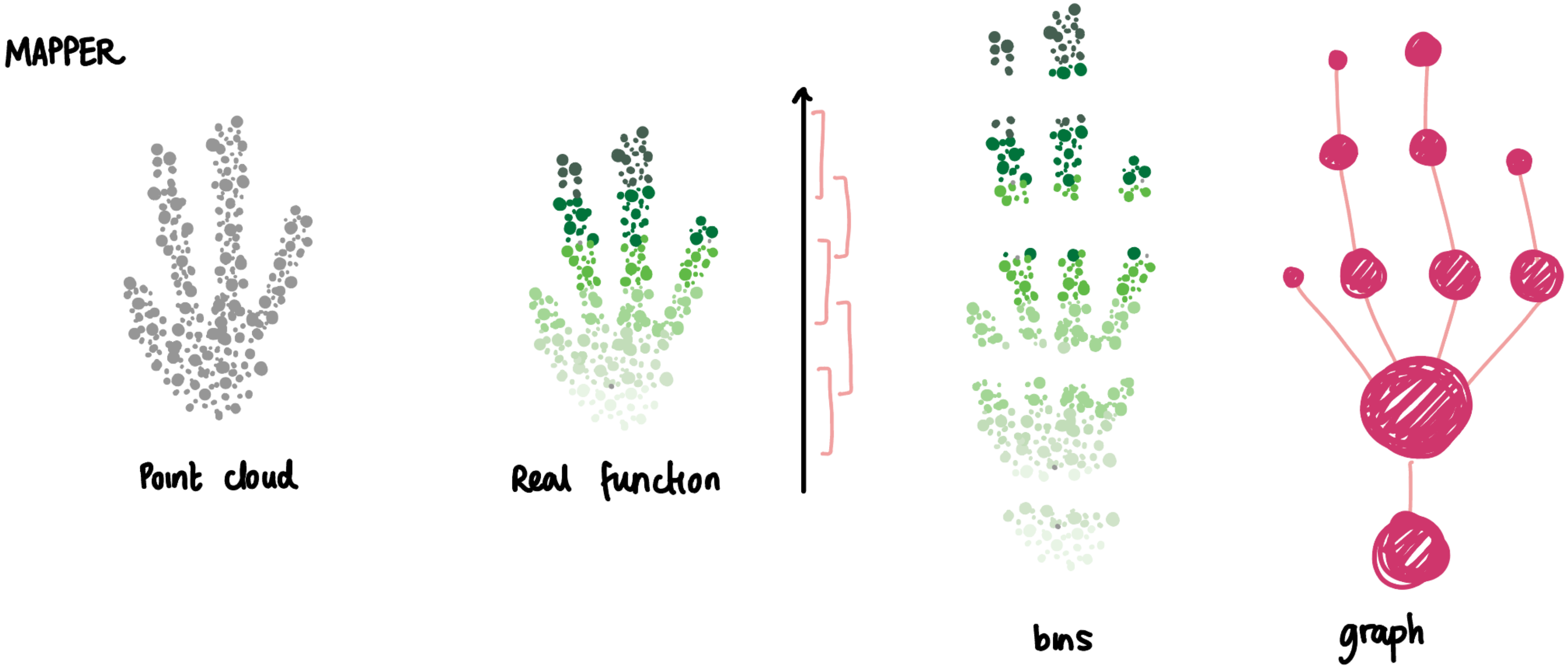
*in dynamical systems [Bozko et al., 2007] [Mrozek, 2017]



** near-linear on graphs [Dey, Hou 2021]



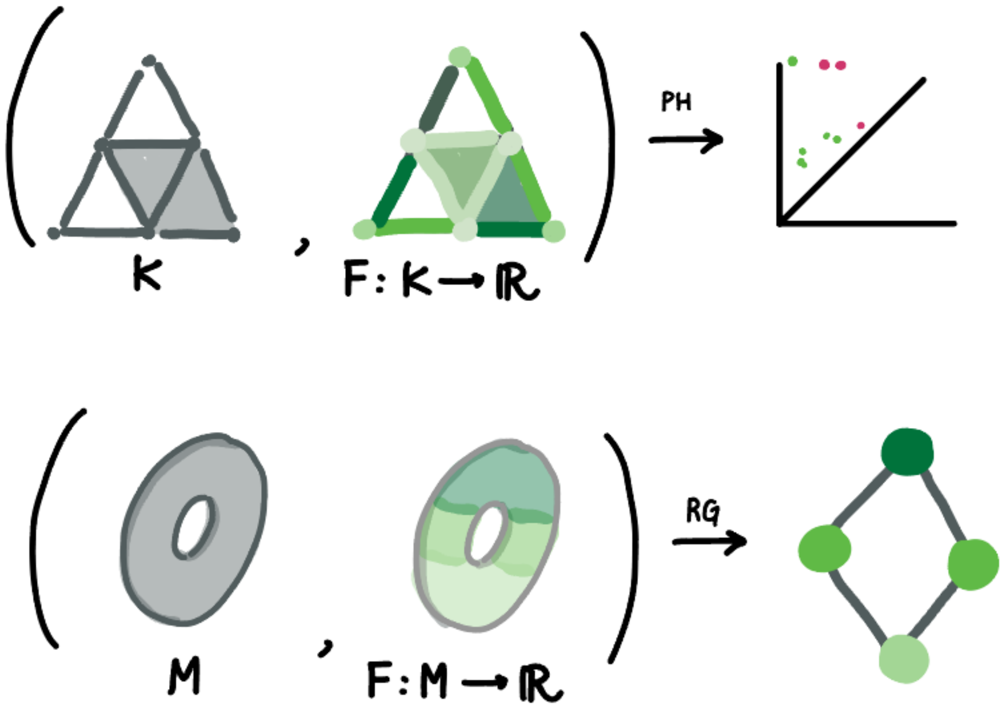
Mapper



[Dey Memoli Wang (2017) *Topological analysis of nerves, Reeb spaces, mappers, and multiscale mappers*]

Reeb Graphs (contour trees in cluster analysis)

Mapper is an algorithm related to Reeb Graphs. These are combinatorial objects used to study a real-valued function defined on a manifold.



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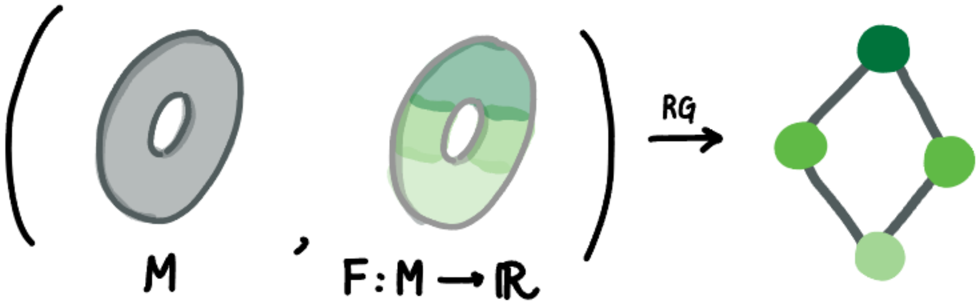

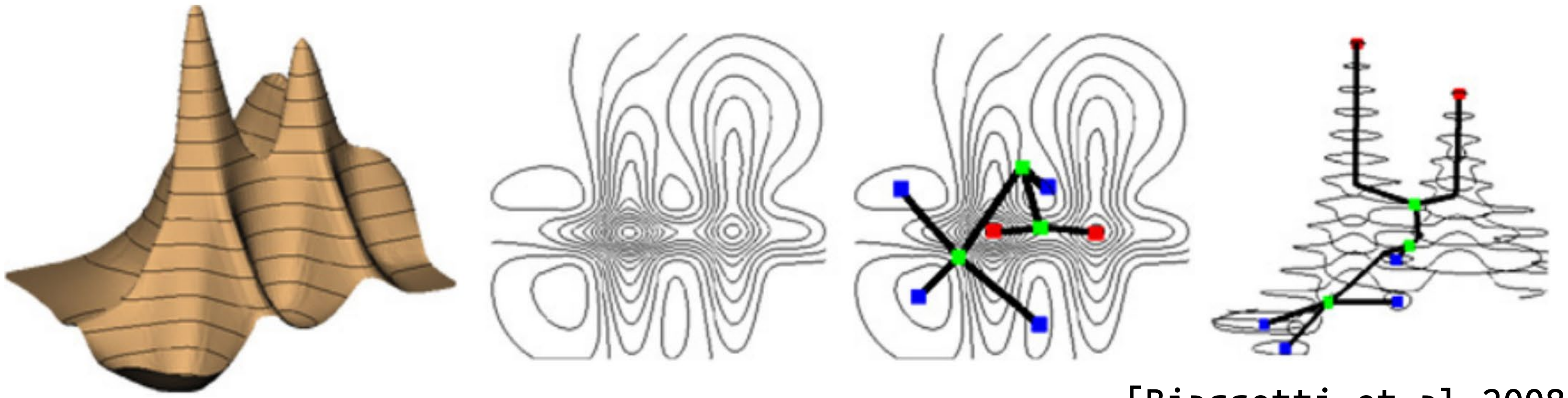

[Biassotti et al 2008]

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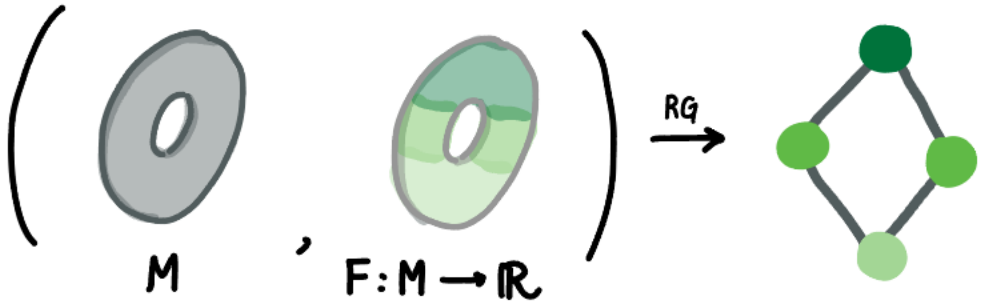

A lot of the research on applications of Reeb Graphs is around finding ways to compare them with one another.

- Interleaving distance `[Morozov et al. 2013]`
- Functional Distortion distance `[Bauer et al. 2014]`
- Edit distance `[Di Fabio, Landi 2016]`
- Bottleneck distance `[Cohen et al. 2006]`

Reeb Graphs (contour trees in cluster analysis)

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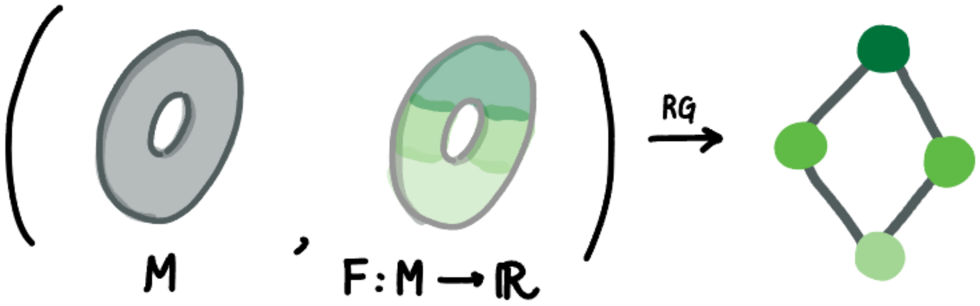


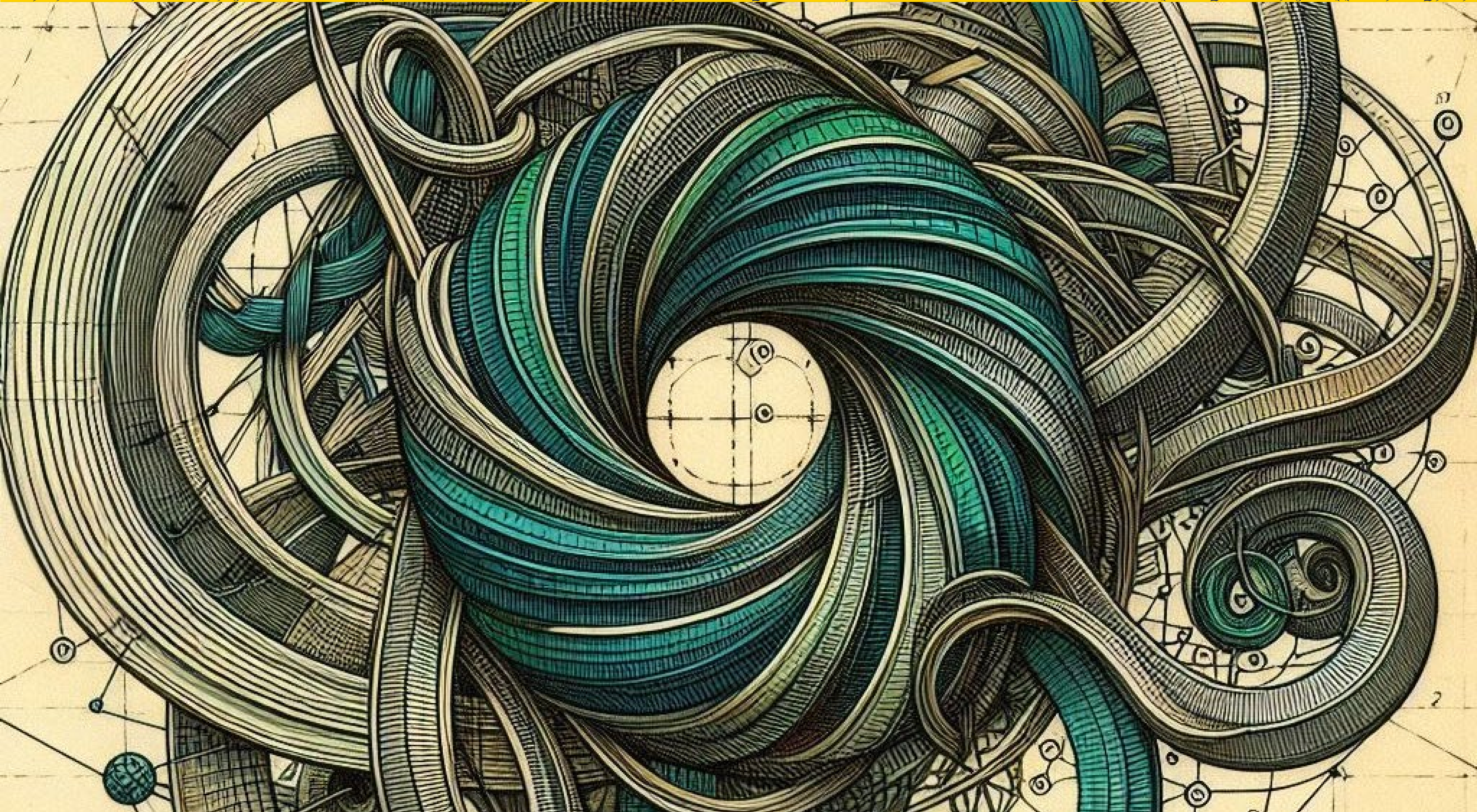
To compute the Reeb graph of a **simplicial complex**, the best approach is to use PL-functions in O(m log m). [Parsa 2013]

Why should we care?

As a community we are starting to look at Mapper to study networks
Maybe looking at how its continuous counterpart has been used might help us.

[Hajij et al 2018]
[Rosen et al 2018]
[Bodnar et al 2020]
[Patania et al 2023ish]

at the heart of the torus, they discovered it: a prime number sequence etched in shimmering silver. The walls trembled, revealing a hidden passage. As they stepped through, the

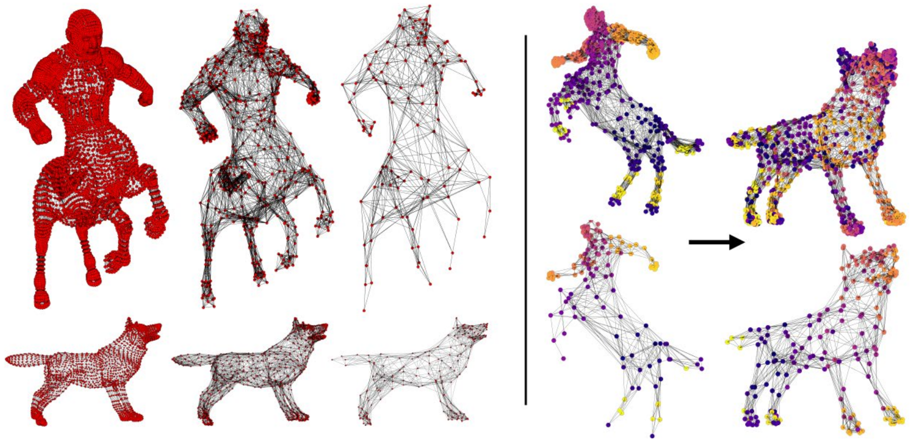


Esoteric Mapper

We have always seen Mapper as a graph, but it can be defined as the skeleton of more general functions and covers of a point cloud, turning it into a proper simplicial complex. `[Singh et al. 2007]`

`In 2016, Dey et al.` created the mythological monster that is **multiscale mapper** putting together ideas from generalized persistence with that of combinatorial mapper. Unaware of its existence, `Chowdhury et al in 2021` used it with co-optimal transport to match combinatiorial structures generated at different resolutions.

Multi-resolution Hypergraph Matching



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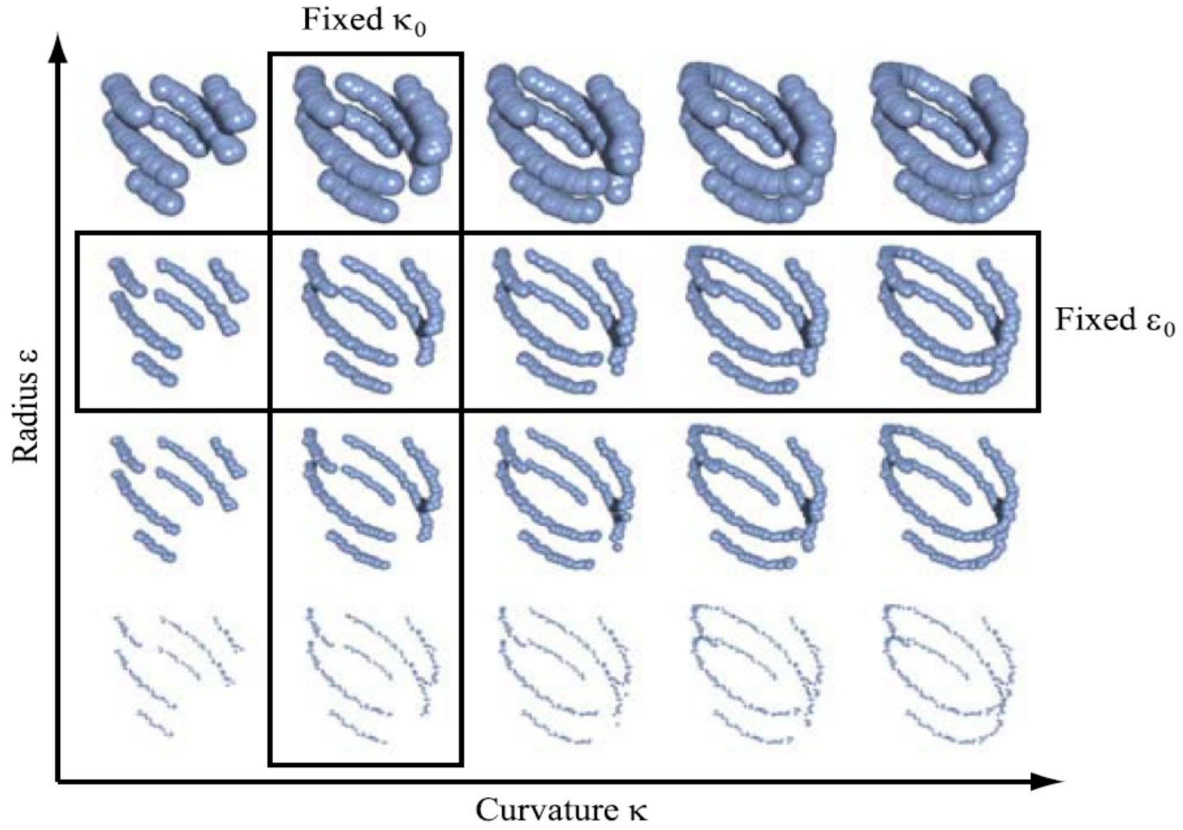
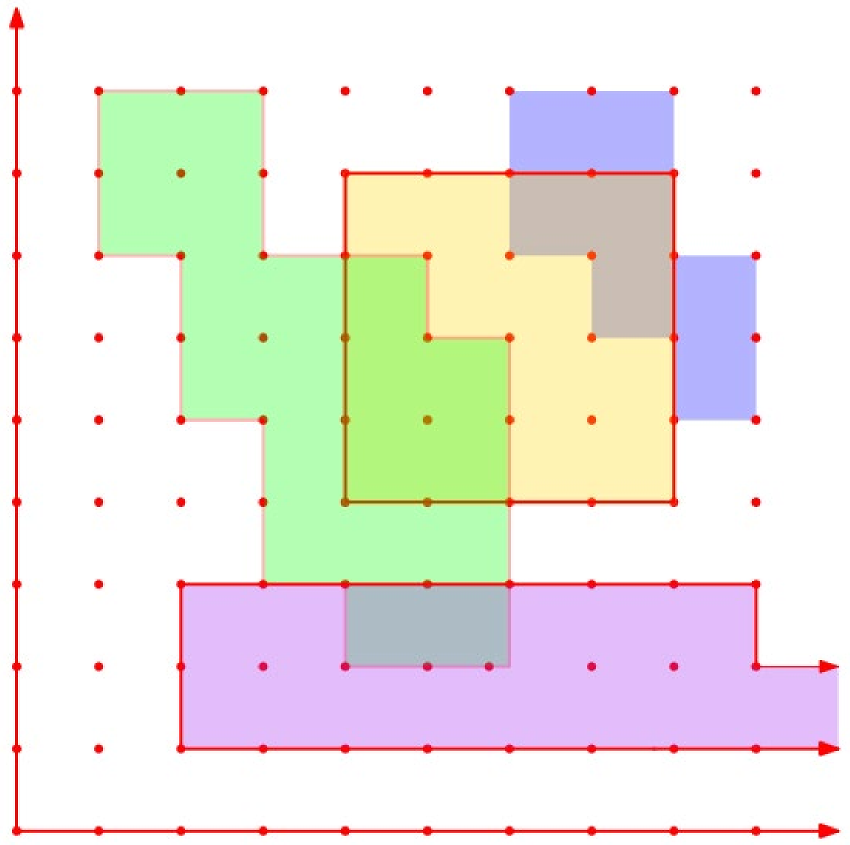
Multiparameter persistence – adding dimensions



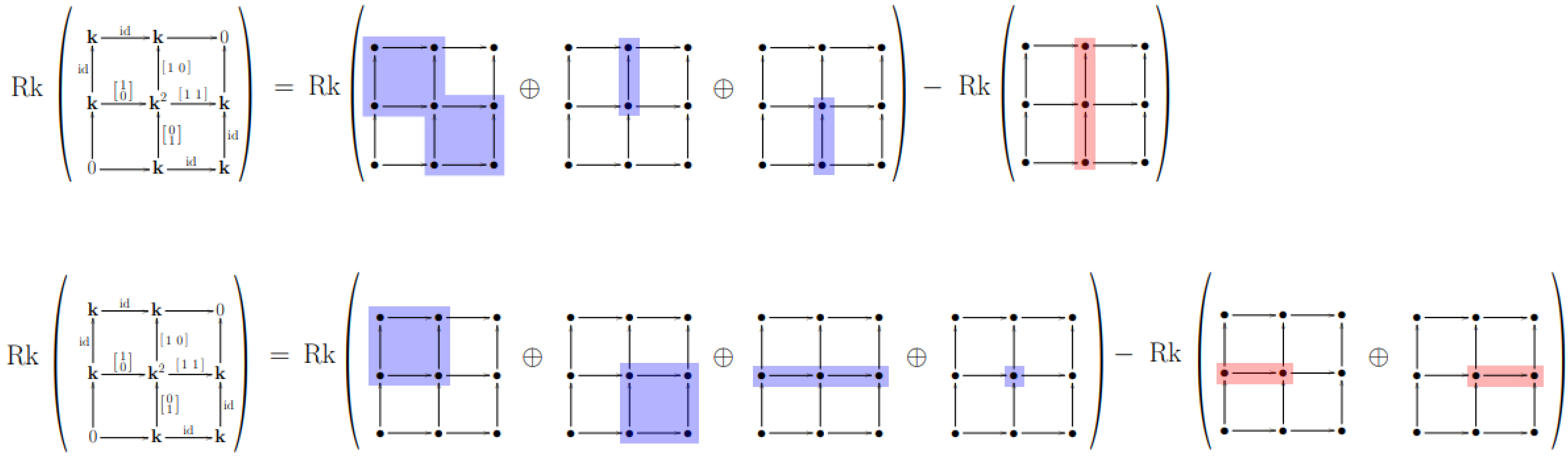
[Carlsson 2009 *The theory of multidimensional persistence*]

Multiparameter persistence – adding one dimension



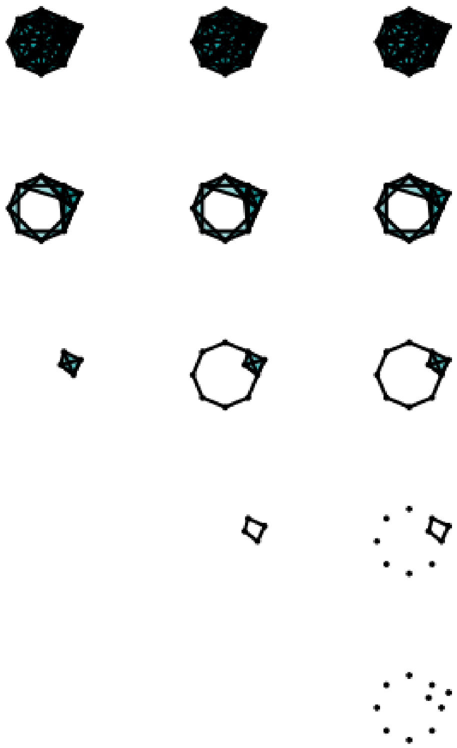
University of Vermont [Bauer et al 2023 *Efficient two-parameter persistence computation via cohomology*]

Multiparameter persistence – adding one dimension



 [Bakke Botnan and Lesnick 2023 *An Introduction to multiparameter persistence*]

Multiparameter persistence – adding one dimension



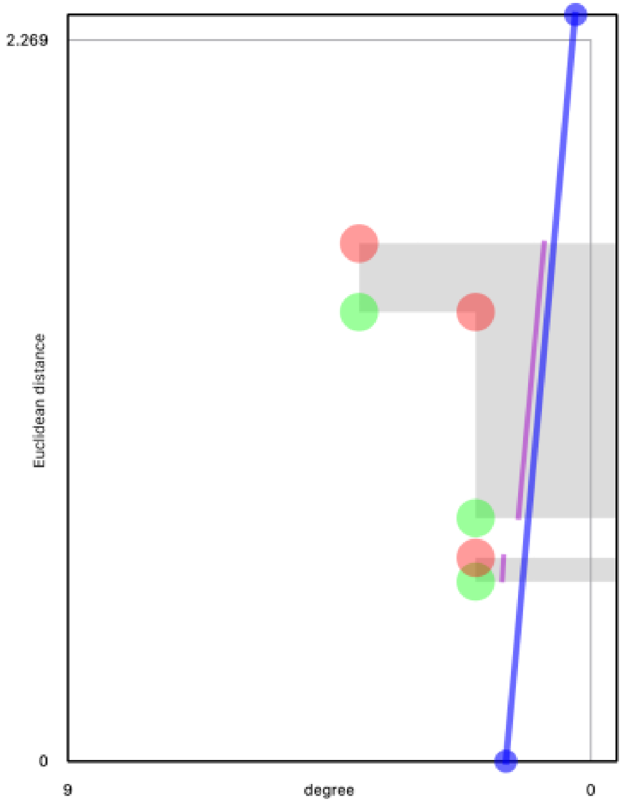
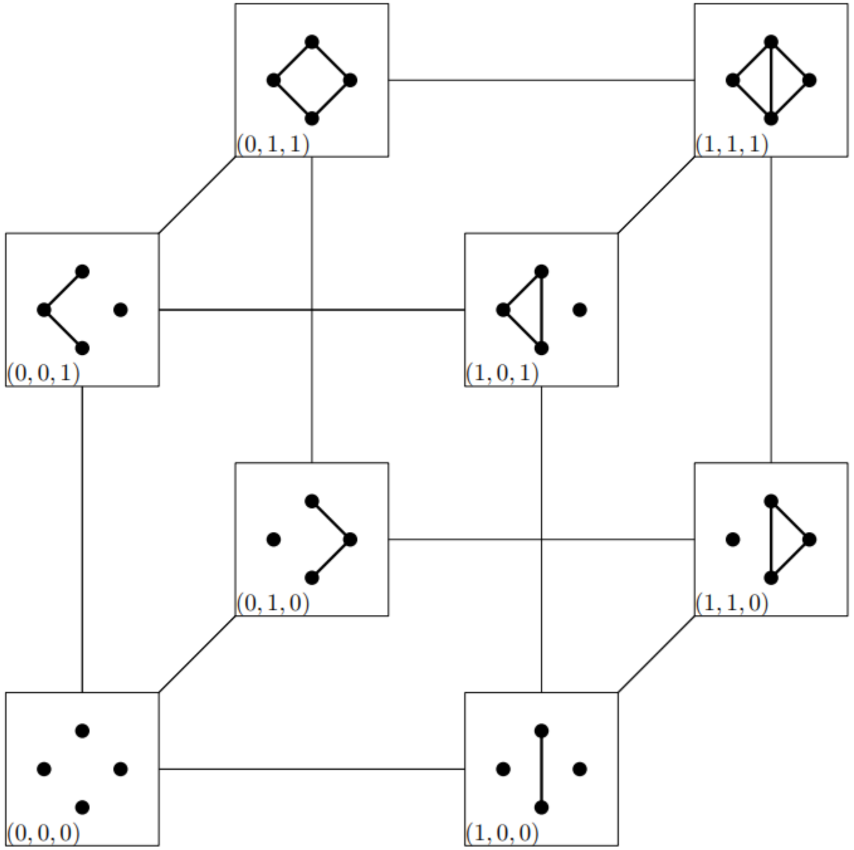


Figure made using RIVET · GitHub

[Bakke Botnan and Lesnick 2023 *An Introduction to multiparameter persistence*]

Multiparameter persistence – adding two dimensions



University of Vermont [Carlsson 2009 *The theory of multidimensional persistence*]

What can you use it for?

What can you use it for?

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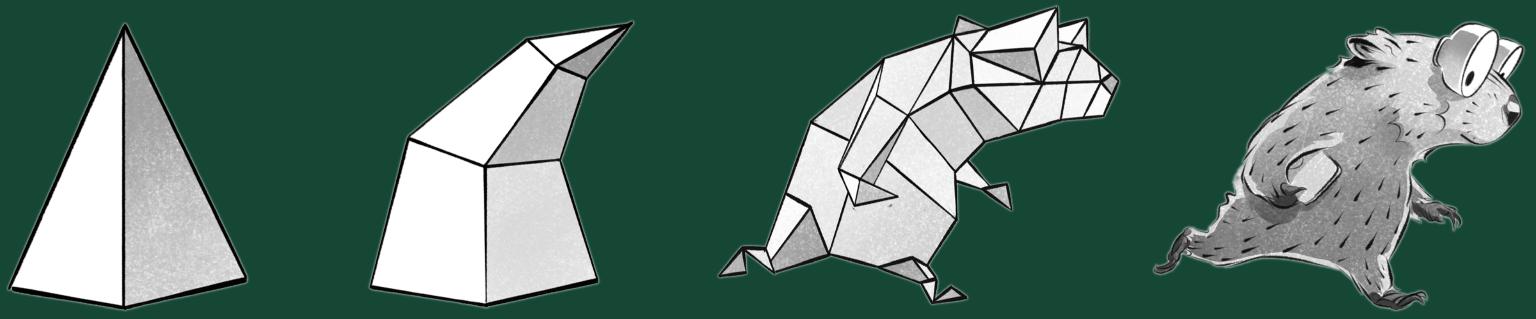
Thank You



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The End