Introduction to TOPOLOGICAL DATA ANALYSIS

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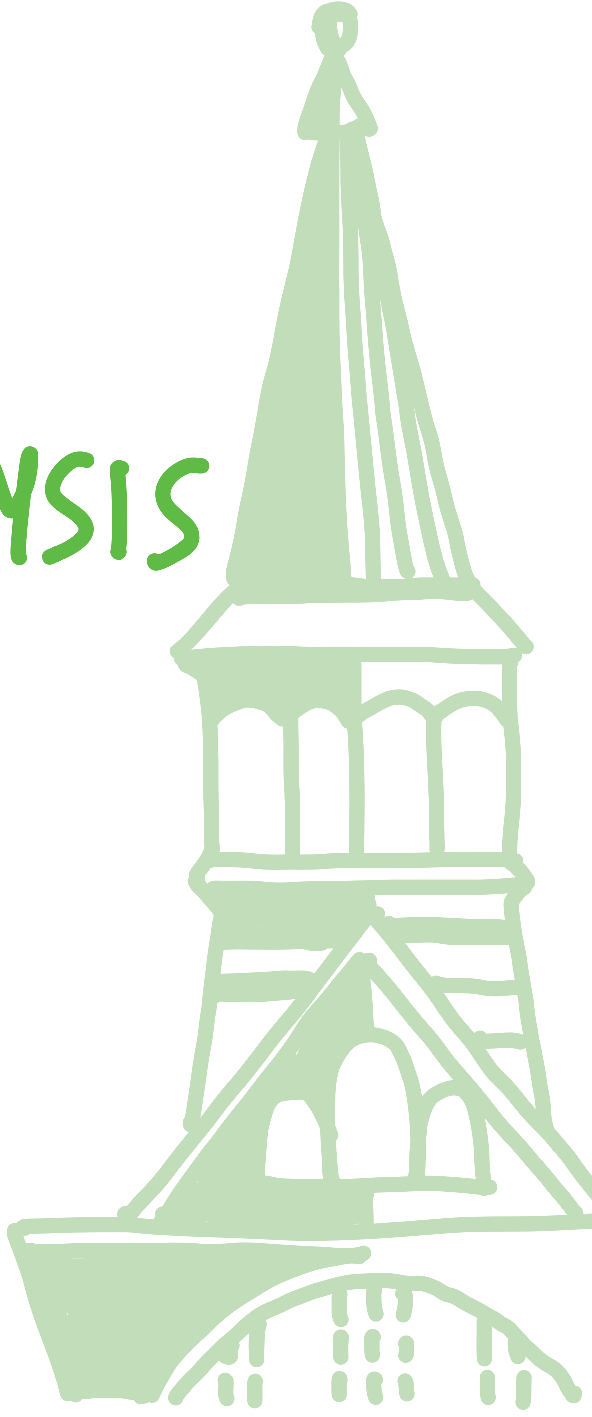


The theory

data	complexes
	filtrations
structure	persistence modules
	homology and barcodes
function	sheaves
	sheaf cohomology

The application

brain data

Alzheimer disease

brain dynamics

What is data?

Let X be a topological space

X is unknown, we want to study it, but we can only collect discrete random samples

What is data?

Let X be a topological space

X is unknown, we want to study it, but we can only collect discrete random samples

data set $D \subseteq X$

What can we learn about X from D? and how can do we do it without bias?

What do I mean about Bias?

when the theoretical model is unknown, coordinates are not necessarly meaningful

similarly, metrics in data sets are not always justified

choosing a parameter gives a partial view

Topology is the cure to all evil!

(IRONY)

when the theoretical model is unknown, coordinates are not necessarly meaningful

coordinate-free method

similarly, metrics in data sets are not always justified

ignore the quantitative values of distance function and get information only the nearness of points

choosing a parameter gives a partial view

construct summaries of information over whole domains of parameters

On the local behaviour of spaces of natural images

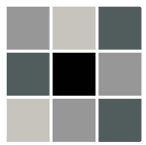
G. Carlsson, T. Ishkhanov, V. de Silva, A. Zomorodian (2008)

Data: 3x3 patches from images

They can be seen as points in 9-dimensional space

normalized to have average = "gray"
norm = 1

$\Rightarrow$ the points lay on a 7-sphere



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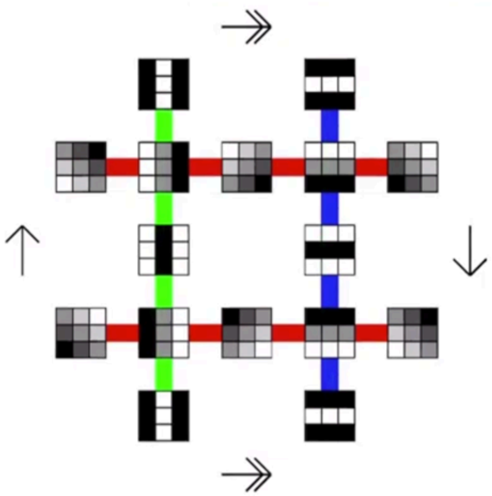
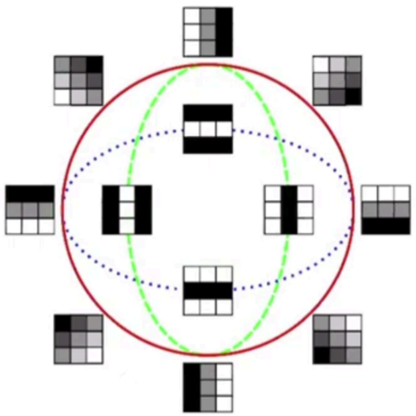
They can be seen as points in 9-dimensional space

$$\left.\begin{array}{l}\text{normalized to have average = "gray"} \\ \text{norm} = 1\end{array}\right\} \Rightarrow \text{the points lay on a 7-sphere}$$

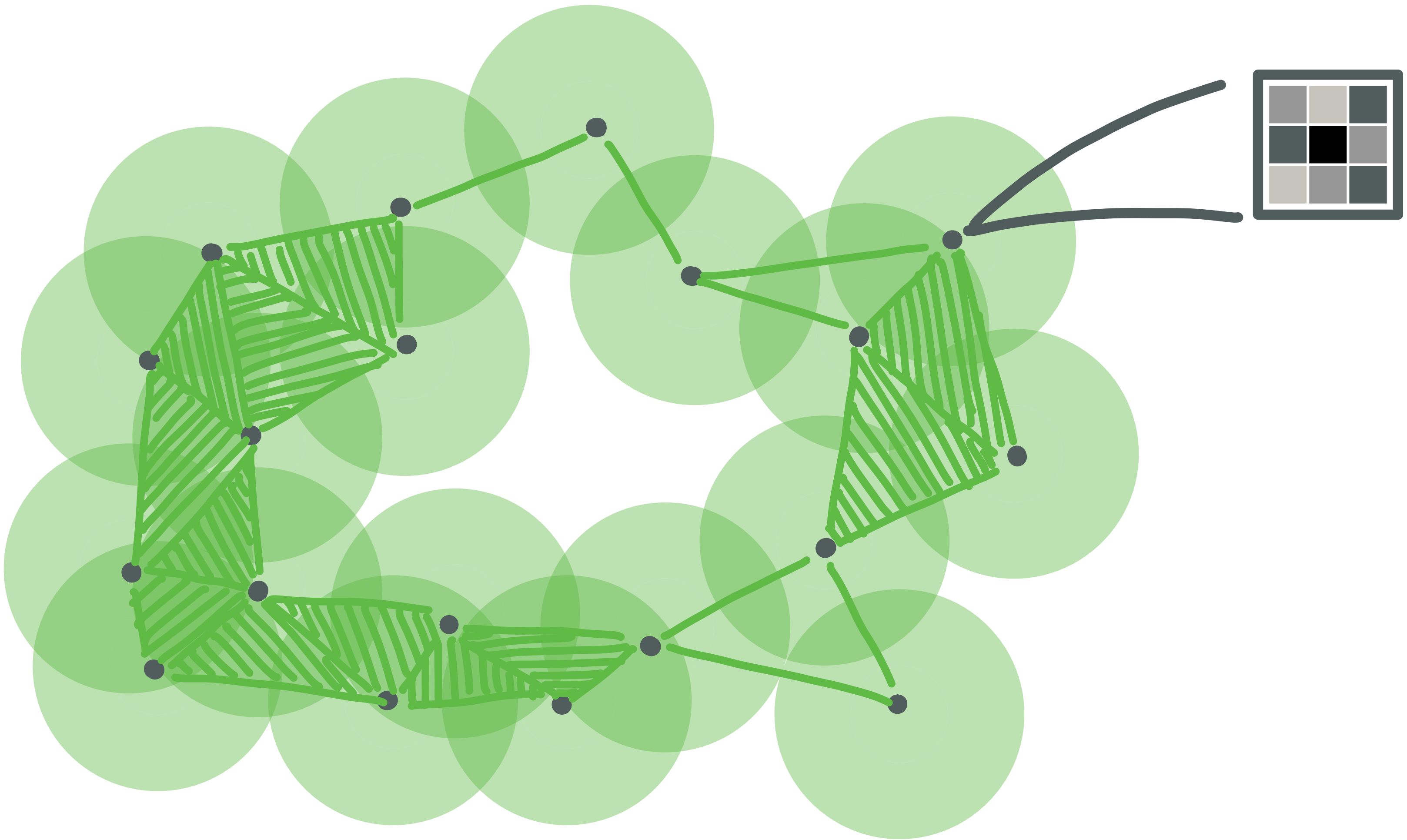
using topological data analysis they were able to find out that the subspace the data comes from has the same homology type as the **KLEIN BOTTLE**





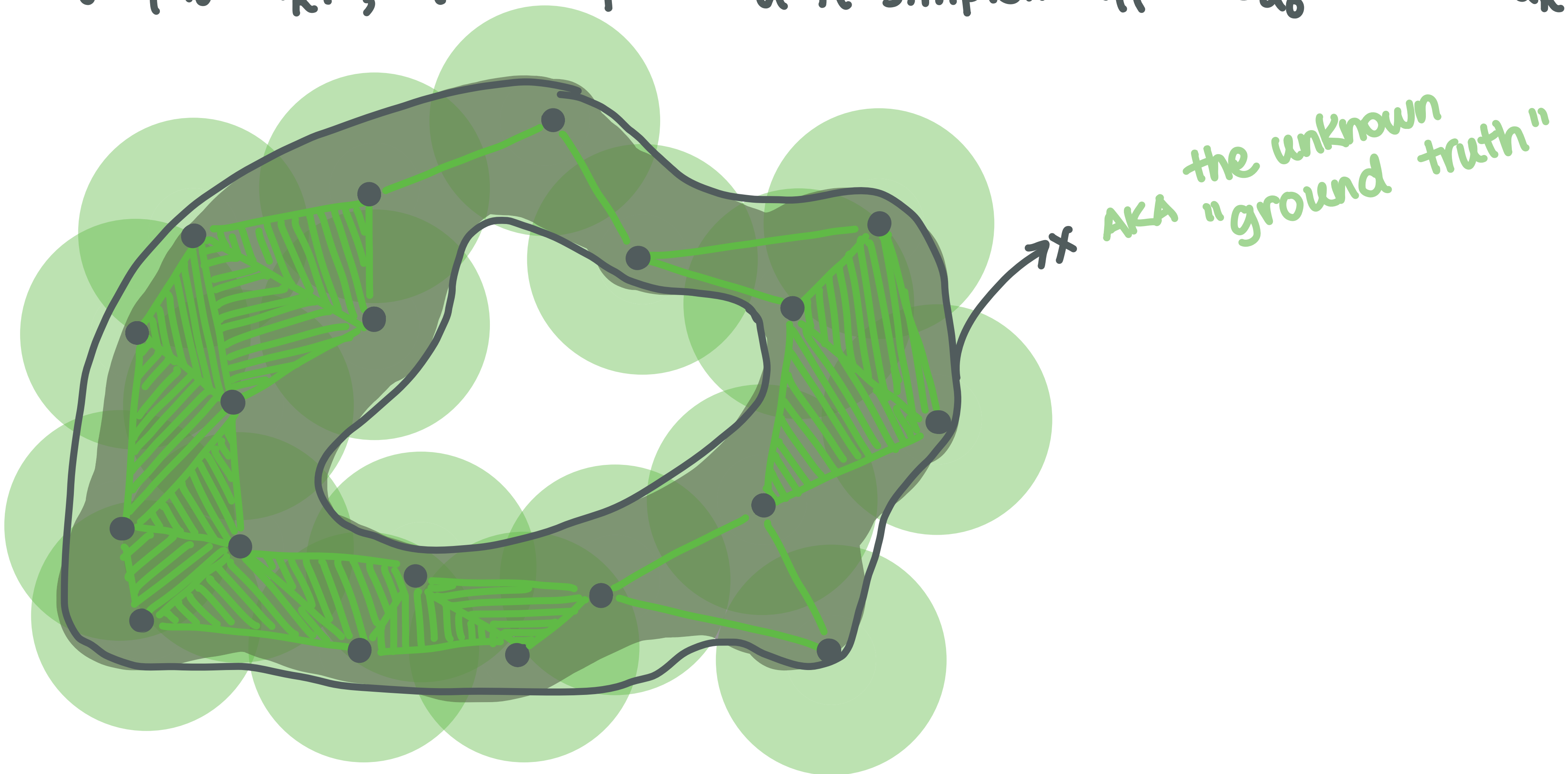
How to go from data to a structure?

In this case the data set comes with an intrinsic metric then we can construct a simplicial complex from it.



Nerve

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ a covering of X.
The nerve of $\mathcal{U}$ is an abstract simplicial complex with vertex set A.
$\sigma = \{a_0 \dots a_k\}$, $a_i \in A$ spans a k-simplex iff $U_{\alpha_0} \cap \dots \cap U_{\alpha_k} \neq \emptyset$



Nerve

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ a covering of X.
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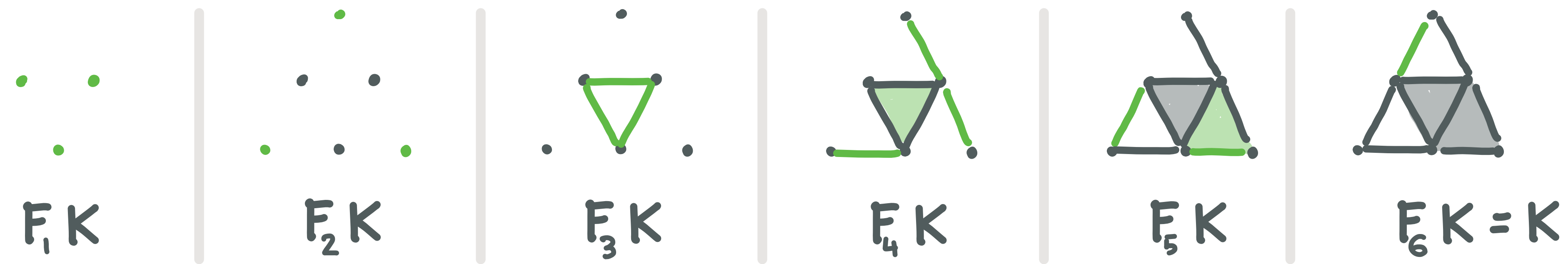
Theorem

Let X be a topological space, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ a countable open covering of X s.t. $\forall S \subseteq A$ with $S \neq \emptyset$ $\bigcap_{s \in S} U_s$ is either contractible or empty.
Then $N(\mathcal{U})$ is homotopy equivalent to X.

Let K be a simplicial complex. A filtration F is a nested sequence of strictly increasing subcomplexes of K $\quad F_1K \subsetneq F_2K \subsetneq \dots \subsetneq F_nK = K$

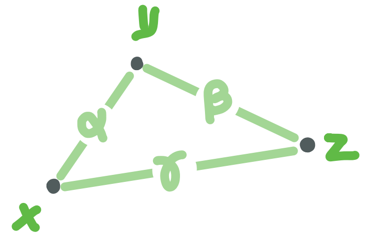


Vietoris-Rips filtration

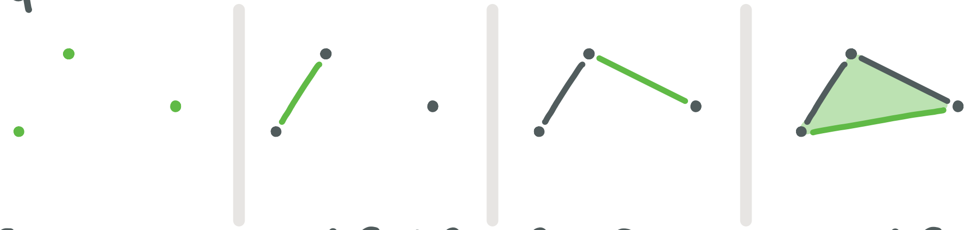
$VR(X,\varepsilon)$ is a simplicial complex

$\sigma = \{a_0 \dots a_k\}$, $a_i \in A$

$\sigma \in K$ iff the distance between every pair of points in σ is at most ε



$0 \leq \alpha \leq \beta \leq \gamma$



$0 < \varepsilon < \alpha$ | $\alpha \leq \varepsilon < \beta$ | $\beta \leq \varepsilon < \gamma$ | $\gamma \leq \varepsilon$

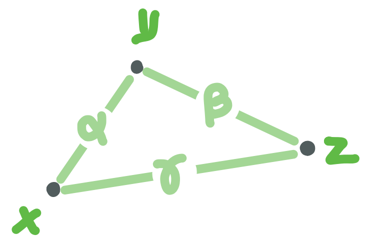
Čech filtration

$\check{C}(X,\varepsilon)$ is a simplicial complex

$\sigma = \{a_0 \dots a_k\}$, $a_i \in A$

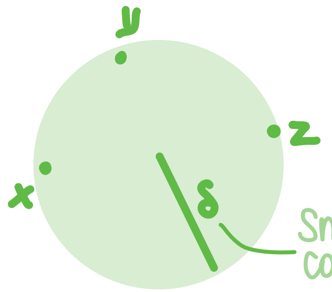
$\sigma \in K$ iff $B^{(\varepsilon)}_{a_0} \cap \dots \cap B^{(\varepsilon)}_{a_k} \neq \emptyset$

$B^{(\varepsilon)}_x$ = ball centered in x of radius ε



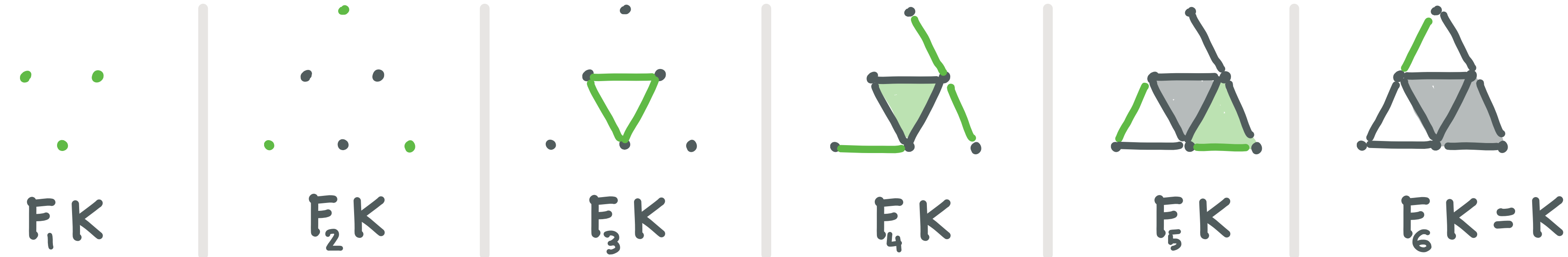
$0 \leq \alpha \leq \beta \leq \gamma$

$0 < \varepsilon < \frac{\alpha}{2}$ | $\frac{\alpha}{2} \leq \varepsilon < \frac{\beta}{2}$ | $\frac{\beta}{2} \leq \varepsilon < \frac{\gamma}{2}$ | $\frac{\gamma}{2} \leq \varepsilon < \delta$ | $\delta < \varepsilon$



Smallest radius ball that contains all three points

Let K be a simplicial complex. A filtration F is a nested sequence of strictly increasing subcomplexes of K $F_1K \subsetneq F_2K \subsetneq \dots \subsetneq F_nK = K$

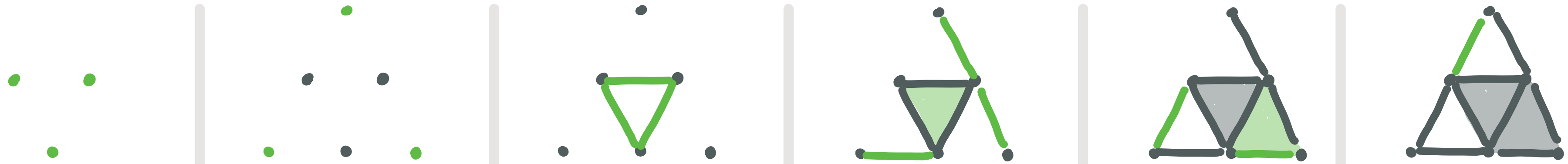


Functoriality of Homology

- If $f: K \to L$ and $g: L \to M$ simplicial maps then $C_\cdot(g \circ f) = C_\cdot(g) \circ C(f)$
- If $\varphi: (C,d) \to (C',d')$ and $\psi: (C',d') \to (C'',d'')$ chain maps then

$$H_\cdot(\psi \circ \varphi) = H_\cdot(\psi) \circ H_\cdot(\varphi)$$

Let K be a simplicial complex. A **filtration** F is a nested sequence of strictly increasing subcomplexes of K $\quad F_1K \subsetneq F_2K \subsetneq \dots \subsetneq F_nK = K$



$$\begin{tikzcd}
F_1K \arrow[r, hook] \arrow[d] & F_2K \arrow[r, hook] \arrow[d] & F_3K \arrow[r, hook] \arrow[d] & F_4K \arrow[r, hook] \arrow[d] & F_5K \arrow[r, hook] \arrow[d] & F_6K = K \arrow[d] \\
H_\bullet(F_1K) \arrow[r] & H_\bullet(F_2K) \arrow[r] & H_\bullet(F_3K) \arrow[r] & H_\bullet(F_4K) \arrow[r] & H_\bullet(F_5K) \arrow[r] & H_\bullet(K)
\end{tikzcd}$$

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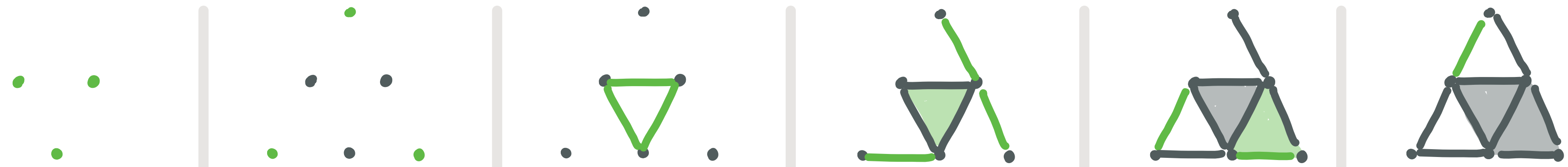

$$
\begin{tikzcd}
F_1K \arrow[r, hook] \arrow[d] & F_2K \arrow[r, hook] \arrow[d] & F_3K \arrow[r, hook] \arrow[d] & F_4K \arrow[r, hook] \arrow[d] & F_5K \arrow[r, hook] \arrow[d] & F_6K = K \arrow[d] \\
H_\bullet(F_1K) \arrow[r] & H_\bullet(F_2K) \arrow[r] & H_\bullet(F_3K) \arrow[r] & H_\bullet(F_4K) \arrow[r] & H_\bullet(F_5K) \arrow[r] & H_\bullet(K)
\end{tikzcd}
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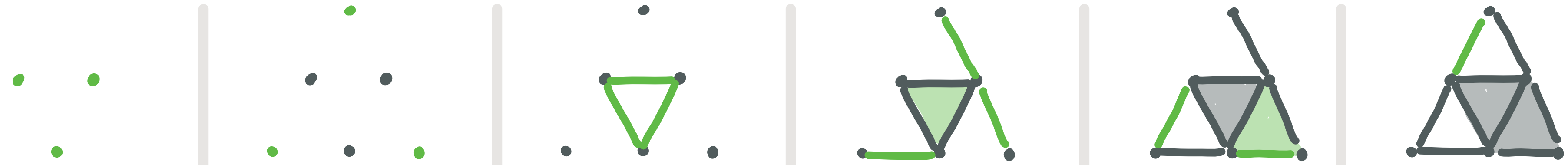
PERSISTENT HOMOLOGY

Let K be a simplicial complex. A filtration F is a nested sequence of strictly increasing subcomplexes of K $\quad F_1K \subsetneq F_2K \subsetneq \dots \subsetneq F_nK = K$



$$
\begin{tikzcd}
F_1K \arrow[r, hook] \arrow[d] & F_2K \arrow[r, hook] \arrow[d] & F_3K \arrow[r, hook] \arrow[d] & F_4K \arrow[r, hook] \arrow[d] & F_5K \arrow[r, hook] \arrow[d] & F_6K = K \arrow[d] \\
H_\cdot(F_1K) \arrow[r] & H_\cdot(F_2K) \arrow[r] & H_\cdot(F_3K) \arrow[r] & H_\cdot(F_4K) \arrow[r] & H_\cdot(F_5K) \arrow[r] & H_\cdot(K)
\end{tikzcd}
$$

PERSISTENT HOMOLOGY

persistence homology module

A (discrete) **persistence module** is a pair $(V_\bullet, a_\bullet)$ of vector spaces $\{V_i : i \in \mathbb{N}\}$ and linear maps $a_i : V_i \to V_{i+1}$

$$V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_i \longrightarrow V_{i+1} \longrightarrow \cdots$$

Warning! This is not necessarly a complex (i.e. $a_i \circ a_{i+1}$ is not always $=0$)

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Persistence modules form a category!

A morphism of persistence modules $\varphi : (V_\bullet, a_\bullet) \longrightarrow (W_\bullet, b_\bullet)$

$$\begin{tikzcd}
V_0 \arrow[r, "a_0"] \arrow[d, "\varphi_0"'] & V_1 \arrow[r, "a_1"] \arrow[d, "\varphi_1"'] & \cdots \arrow[r] & V_i \arrow[r, "a_i"] \arrow[d, "\varphi_i"'] & V_{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"'] & \cdots \\
W_0 \arrow[r, "b_0"] & W_1 \arrow[r, "b_1"] & \cdots \arrow[r] & W_i \arrow[r, "b_i"] & W_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

If $\forall i$ φ_i is invertible, then φ is an **isomorphism**

The **direct sum** of persistence modules $(V_\bullet, a_\bullet) \oplus (W_\bullet, b_\bullet) = (V_\bullet \oplus W_\bullet, a_\bullet \oplus b_\bullet)$

$$V_0 \oplus W_0 \xrightarrow[a_0 \oplus b_0]{} V_1 \oplus W_1 \xrightarrow[a_1 \oplus b_1]{} \cdots \longrightarrow V_i \oplus W_i \xrightarrow[a_i \oplus b_i]{} \cdots$$

A persistence module is **indecomposable** if it admits no interesting direct sum decomposition

if $(I,c) \simeq (V,a) \oplus (W,b)$ then $\begin{cases} (V,a) \simeq (I,c) \\ (W,b) \simeq (0,0) \end{cases}$

isomorphism

Property

Persistence modules of the following type are indecomposable for $i \leq j$ with $i,j \in \mathbb{N}$

$$
\begin{tikzcd}
I_0 \arrow[r] \arrow[d, equal] & I_1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & I_{i-1} \arrow[r] \arrow[d, equal] & I_i \arrow[r] \arrow[d, equal] & \cdots \quad \cdots \arrow[r] & I_j \arrow[r] \arrow[d, equal] & I_{j+1} \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{F} \arrow[r, "id_{\mathbb{F}}"'] & \cdots \arrow[r, "id_{\mathbb{F}}"'] & \mathbb{F} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots
\end{tikzcd}
$$

field

identity map

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if $(I,c) \underset{\text{isomorphism}}{\simeq} (V,a) \oplus (W,b)$ then $\begin{cases} (V,a) \simeq (I,c) \\ (W,b) \simeq (0,0) \end{cases}$

Property

Persistence modules of the following type are indecomposable for $i \leq j$ with $i,j \in \mathbb{N}$

$$\underbrace{\begin{array}{ccccccccccccccccc}
I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots & \longrightarrow & I_{i-1} & \longrightarrow & I_i & \longrightarrow & \cdots & \longrightarrow & I_j & \longrightarrow & I_{j+1} & \longrightarrow & \cdots \\
\| & & \| & & & & \| & & \| & & & & \| & & \| & & \\
0 & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \cdots & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \mathbb{F} & \xrightarrow[\mathrm{id}_{\mathbb{F}}]{} & \cdots & \xrightarrow[\mathrm{id}_{\mathbb{F}}]{} & \mathbb{F} & \xrightarrow[0]{} & 0 & \xrightarrow[0]{} & \cdots
\end{array}}_{(I^{i,j},\, c^{i,j}) \quad \text{Interval } [i,j] \text{ module}}$$

Decomposition Theorem

$\forall i \quad \dim V_i < \infty$ and $a_j : V_j \to V_{j+1}$
a_j is isomorphism $\forall j >> 0$

Let $(V_\bullet, a_\bullet)$ be a finite type persistence module then exists a finite set of intervals $Bar(V_\bullet, a_\bullet) = \{ [i,j] \ \ i \in \mathbb{N}, j \in \mathbb{N} \cup \{\infty\} \ \ j \geq i \}$ and a multiplicity $\mu : Bar(V_\bullet, a_\bullet) \longrightarrow \mathbb{N}$ so that:

$$\underbrace{(V_\bullet, a_\bullet) \simeq \bigoplus_{[i,j] \in Bar(V,a)} (I^{i,j}, c^{i,j})^{\mu[i,j]}}_{\text{Barcode decomposition}}$$

(idea)
PROOF uses the classification of freely generated $\mathbb{F}[t]$-modules into a free and torsion part. Because there is a lemma saying that we can always represent a persistence tame module as a graded module over $\mathbb{F}[t]$

Decomposition Theorem

(or tame)
$\forall i \ \dim V_i < \infty$ and $a_j : V_j \to V_{j+1}$
a_j is isomorphism $\forall j >> 0$

Let $(V_\bullet, a_\bullet)$ be a finite type persistence module then exists a finite set of intervals $Bar(V_\bullet, a_\bullet) = \{ [i,j] \ \ i \in \mathbb{N}, j \in \mathbb{N} \cup \{\infty\} \ \ j \geq i \}$ and a multiplicity $\mu : Bar(V_\bullet, a_\bullet) \longrightarrow \mathbb{N}$ so that:

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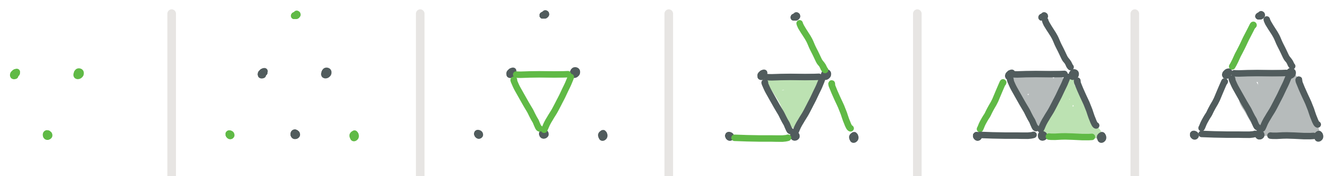
The Barcode decomposition is unique up to reordering of factors



$$F_1 K \hookrightarrow F_2 K \hookrightarrow F_3 K \hookrightarrow F_4 K \hookrightarrow F_5 K \hookrightarrow F_6 K = K$$

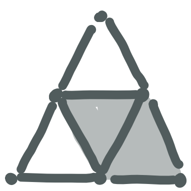
$H_1(F_\bullet K)$

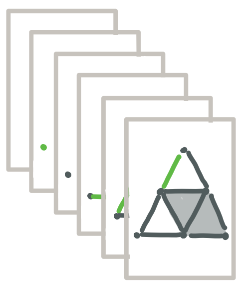
$$0 \to 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to \mathbb{F}^2$$

Persistent Homology

[Simplicial complexes K] $\longrightarrow$ [Filtrations $F_\bullet K$] $\longrightarrow$ [Persistence Modules $H_k(F_\bullet K)$] $\longrightarrow$ [Barcodes $\bigoplus [i,j]^{\mu(i,j)}$]

example





$0 \to 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to \mathbb{F}^2$

$F_1K \hookrightarrow F_2K \hookrightarrow F_3K \hookrightarrow F_4K \hookrightarrow F_5K \hookrightarrow F_6K = K$

$H_\bullet(F_1K) \to H_\bullet(F_2K) \to H_\bullet(F_3K) \to H_\bullet(F_4K) \to H_\bullet(F_5K) \longrightarrow H_\bullet(K)$

The theory

data
- complexes
- filtrations

structure
- persistence modules
- homology and barcodes

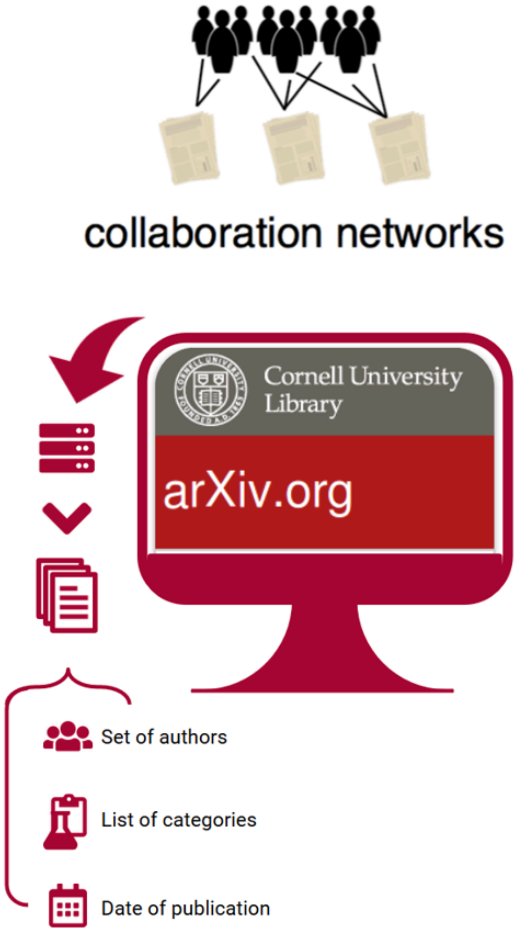
YOU ARE HERE!

function
- sheaves
- sheaf cohomology

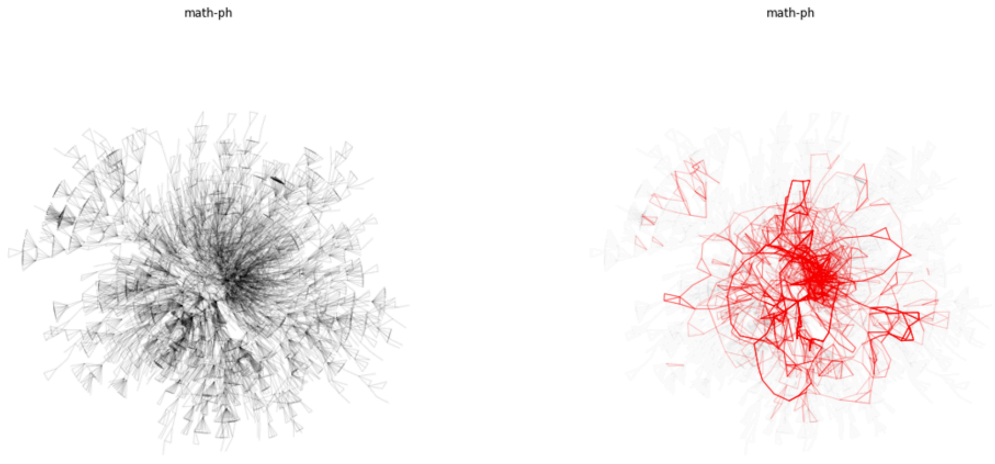
The application

- brain data
- Alzheimer disease
- brain dynamics

Applications



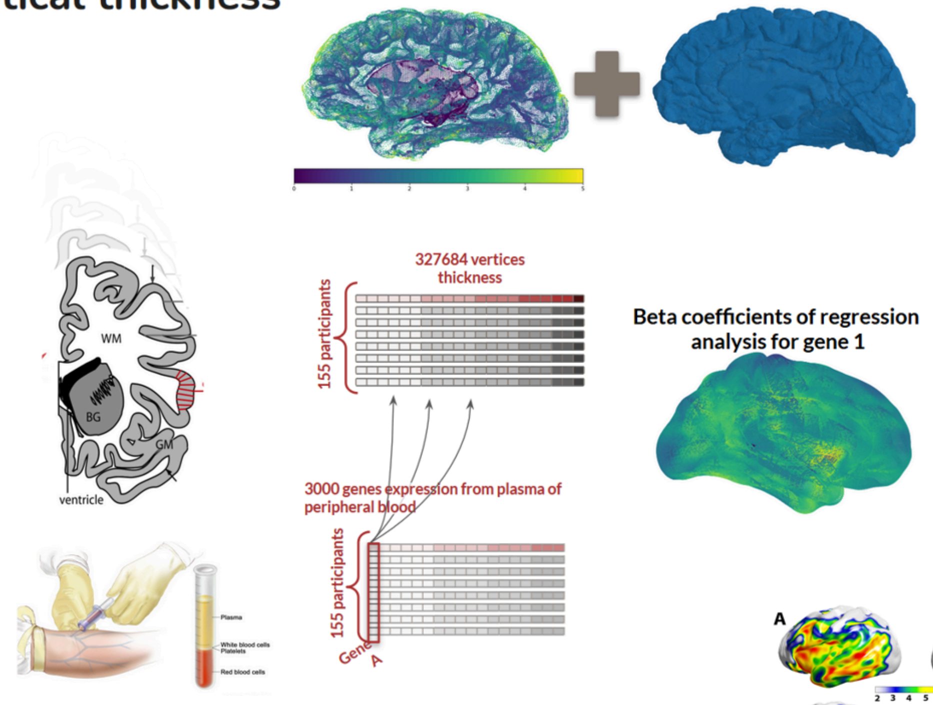
The shape of collaboration
A Patania, F Vaccarino, G Petri - EPJ Data Science (2017)



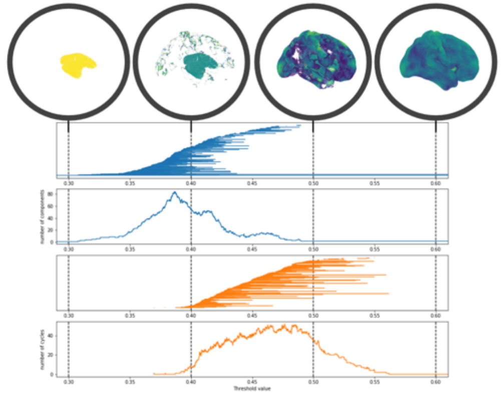
Examples for the biggest connected component for each group.

Applications

Cortical thickness

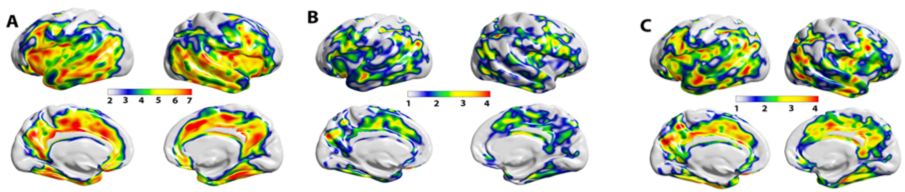


Persistent Homology



National Institutes of Health

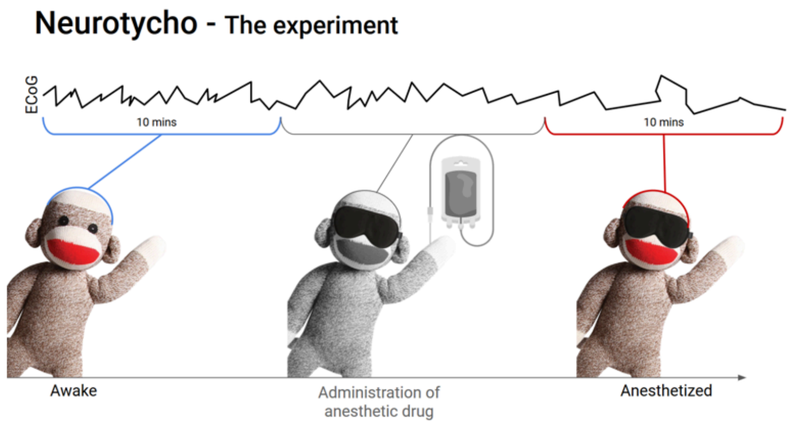
R21 - **Integrative Predictive Modeling of Alzheimer's Disease**
NIH Exploratory/Developmental Research Grant

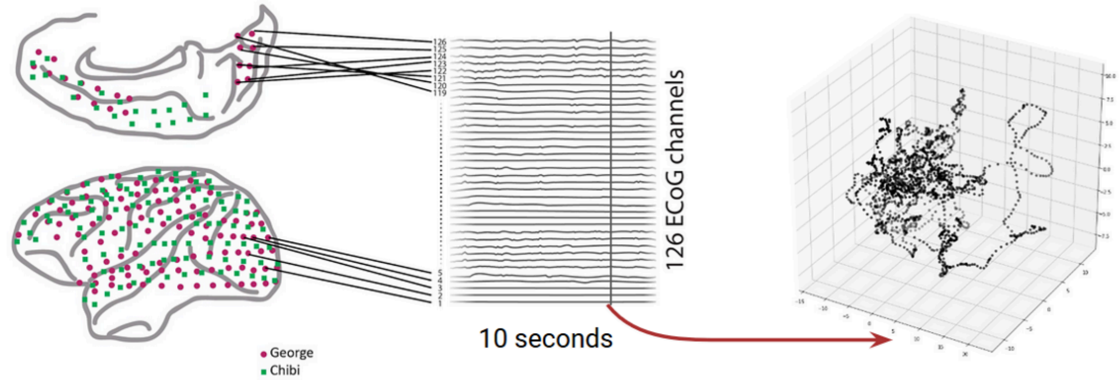


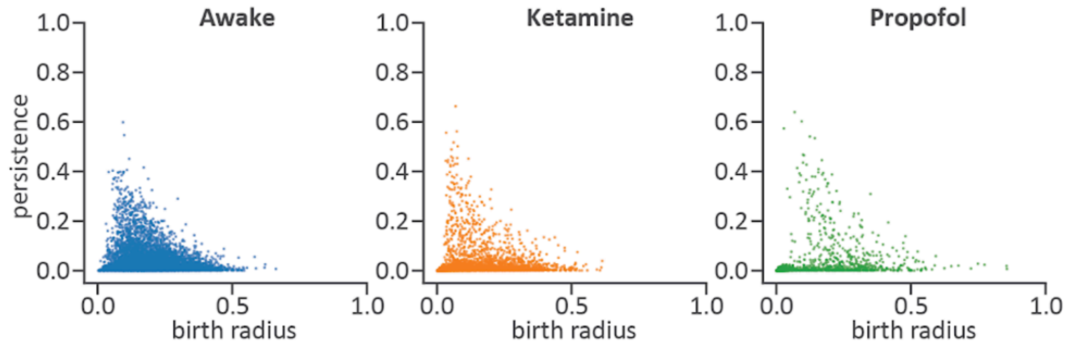
Cluster association(T-statistic) with cortical thickness($T>1.96$, $p<0.05$)
(A) Cluster 5 (B) Cluster 14 (C) Cluster 20

Characterization of genetic expression patterns in Mild Cognitive Impairment using a multiomics approach and neuroimaging endophenotypes
A Bharthur Sanjay, **A Patania**, X Yan, D Svaldi, T Duran, N Shah, E Chen, LG Apostolova (2021)

Applications







Topological Analysis of Differential Effects of Ketamine and Propofol Anesthesia on Brain Dynamics
T F. Varley, V Denny, O Sporns, **A Patania** (2021) in submission Open Science (bioarxiv- https://doi.org/10.1101/2020.04.04.025437)

A **simplicial map** $f: K \longrightarrow L$, $\sigma \longmapsto f(\sigma)$ where $f(\sigma)$ is a simplex of L such that $\dim f(\sigma) \leq \dim(\sigma)$ and if $\dim \sigma = 0$, $\dim f(\sigma) = 0$

Let $f: K \longrightarrow L$ simplicial map, the **fiber** of $\tau \in L$ is the set $\tau/_f$

$\tau/_f = \{a \in K \mid f(a) \subseteq \tau\}$ these are the simplices of K that end up being faces of the simplex τ

Proposition

i) $\tau/_f$ is a subcomplex of K

ii) $\tau/_f \subseteq \tau'/_f$ inclusion of simplicial complexes

Homology is a functor then $\forall \tau \in L$

$$\tau \longmapsto H_k(\tau/_f)$$
$$\tau \leq \tau' \longmapsto H_k(\tau/_f \hookrightarrow \tau'/_f)$$

This is a well known construction in topology: a sheaf!

Sheaves

$$
\begin{bmatrix}\text{Simplicial complexes}\\ K\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Filtrations}\\ F_\bullet K\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Persistence Modules}\\ H_k(F_\bullet K)\end{bmatrix} \longrightarrow \begin{bmatrix}\text{Barcodes}\\ \bigoplus [i,j]^{\mu(i,j)}\end{bmatrix}
$$

$$
\begin{bmatrix}\text{Simplicial complexes}\\ K\end{bmatrix} \dashrightarrow \begin{bmatrix}\text{Simplicial Maps}\\ \varphi : K \to L\end{bmatrix} \dashrightarrow \begin{bmatrix}\text{Sheaf}\\ \mathcal{F}(K)\end{bmatrix} \dashrightarrow \begin{bmatrix}\text{Cohomology}\\ H^k(\mathcal{F}(K))\end{bmatrix}
$$

L as a poset is a category

A **sheaf** on a simplicial complex is a functor $\mathcal{F}: (L, \subseteq) \longrightarrow \mathrm{Vect}_{\mathbb{F}}$

category of vector spaces over a field

$\mathcal{F}$ assigns to each simplex $\tau \in L$ a vector space $\mathcal{F}(\tau)$ called the **stalk** of $\mathcal{F}$ over τ
inclusion $\tau' \subseteq \tau$ a linear map $\mathcal{F}(\tau' \subseteq \tau): \mathcal{F}(\tau') \to \mathcal{F}(\tau)$ **restriction map**

such that the following hold:

identity $\tau = \tau$ is sent to identity map

associativity $\forall \tau \subseteq \tau' \subseteq \tau''$

$$\begin{tikzcd}
\mathcal{F}(\tau) \arrow[r] \arrow[dr] & \mathcal{F}(\tau') \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathcal{F}(\tau'')
\end{tikzcd}$$

diagram commutes

L as a poset is a category

A **sheaf** on a simplicial complex is a **functor** $\mathcal{F} : (L, \subseteq) \longrightarrow \mathrm{Vect}_{\mathbb{F}}$

category of vector spaces over a field

$\mathcal{F}$ assigns to each simplex $\tau \in L$ a vector space $\mathcal{F}(\tau)$ called the **stalk** of $\mathcal{F}$ over τ

inclusion $\tau' \subseteq \tau$ a linear map $\mathcal{F}(\tau' \subseteq \tau) : \mathcal{F}(\tau') \to \mathcal{F}(\tau)$ **restriction map**

such that the following hold:

identity $\quad \tau = \tau$ is sent to identity map

associativity $\quad \forall \tau \subseteq \tau' \subseteq \tau''$

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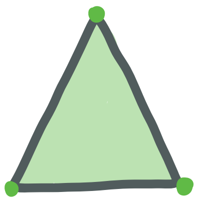
small example

L = 

simplicial complex

$$\mathcal{F} = \begin{tikzcd} & & \mathcal{F}(0) \arrow[dl] \arrow[dd] \arrow[dr] & & \\ & \mathcal{F}(01) \arrow[dr] & & \mathcal{F}(02) \arrow[dl] & \\ & & \mathcal{F}(012) & & \\ \mathcal{F}(1) \arrow[uur] \arrow[rr] & & \mathcal{F}(12) \arrow[u] & & \mathcal{F}(2) \arrow[ll] \arrow[uul] \end{tikzcd}$$

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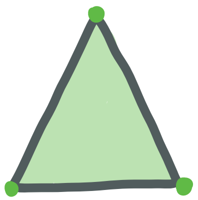
$L =$ 

simplicial complex

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 & & \mathcal{F}(0) \arrow[loop above] \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & \mathcal{F}(01) \arrow[loop left] \arrow[dr] & & \mathcal{F}(02) \arrow[loop right] \arrow[dl] & \\
 & & \mathcal{F}(012) & & \\
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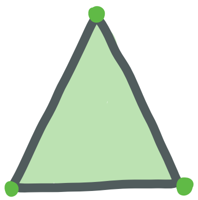

L=

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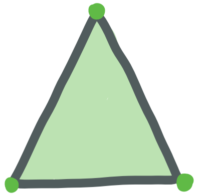
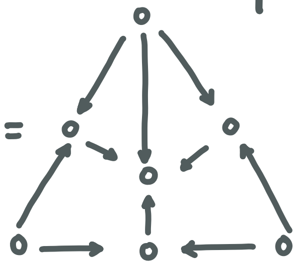
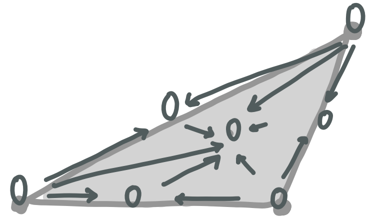
Meaningful sheaves

zero sheaf assigns the 0 vector space to every $\tau \in L$ and all inclusions to 0 map

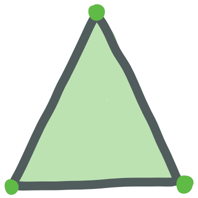
$L =$ 

simplicial complex

$\mathcal{F} =$ 



skyscraper sheaf fix a simplex $\tau \in L$, assigns $\begin{cases} \mathbb{F} \text{ to } \tau \\ 0 \text{ to } \tau' \neq \tau \end{cases}$, all inclusions to 0 map

$L =$ 

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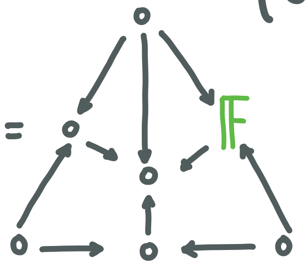
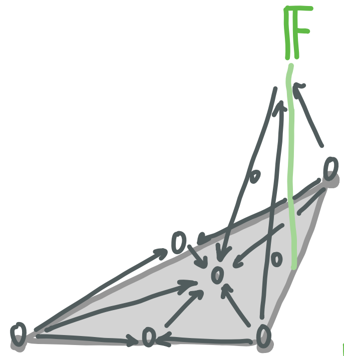
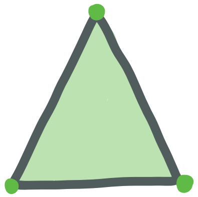
$\mathcal{F} =$ 



constant sheaf $\mathbb{F}_L$ assigns $\mathbb{F}$ to every simplex $\tau \in L$ and $id_{\mathbb{F}}$ to every inclusion $\tau' \leq \tau$

$L =$ 

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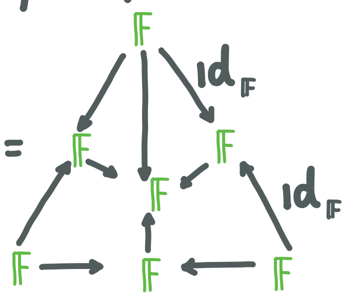
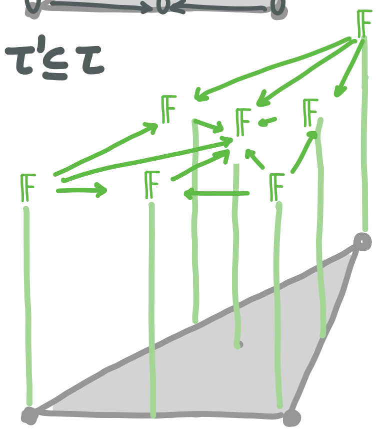
$\mathcal{F} =$ 



Sheaf Cohomology

To define a cohomology, we need to define a cochain complex.
For that we need cochain groups and coboundary maps

cochain group of L with coefficients in the sheaf $\mathcal{S}$ is the vector space

$$C^k(L;\mathcal{S}) = \prod_{\dim \tau = k} \mathcal{S}(\tau)$$

coboundary map $\partial^k_{\mathcal{S}} : C^k(L;\mathcal{S}) \longrightarrow C^{k+1}(L;\mathcal{S})$ is the linear combination $\sum (-1)^i \, \mathcal{S}(\underbrace{\tau}_{\dim k} \leq \underbrace{\tau'}_{\dim k+1})$

Proposition

The sequence $0 \longrightarrow C^0(L,\mathcal{S}) \xrightarrow{\partial^0_{\mathcal{S}}} C^1(L,\mathcal{S}) \xrightarrow{\partial^1_{\mathcal{S}}} \dots$ is a co-chain complex

We can define the sheaf cohomology of L with coefficients in $\mathcal{S}$ as $H^k(L;\mathcal{S}) = \operatorname{Ker} \partial^k / \operatorname{Im} \partial^{k-1}$

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NOTE

When $\mathcal{S} = \mathbb{F}_L$ is the constant sheaf, the sheaf cohomology is the classical simplicial cohomology

Sheaves in the wild

In practice, sheaves can be considered as way to encode very complex data structures without having to build overly complex, overfitted models.

Sheaves are good, for example, to represent time-series, images, and videos.

V-sampling sheaf $\cdots \longleftarrow V \longrightarrow 0 \longleftarrow V \longrightarrow 0 \longleftarrow \cdots$ where V a vector space
(for discrete sampling)

supported on a subset A of L is a sheaf whose stalks are V in each cell in A and 0 everywhere else

H^0 sheaf cohomology can be interpreted as the number of connected solutions / global sections of the system defined by the linear transformations + the simplicial structure

Some interesting applications

opinion dynamics / spectral theory	Robert Ghrist, Jakob Hansen
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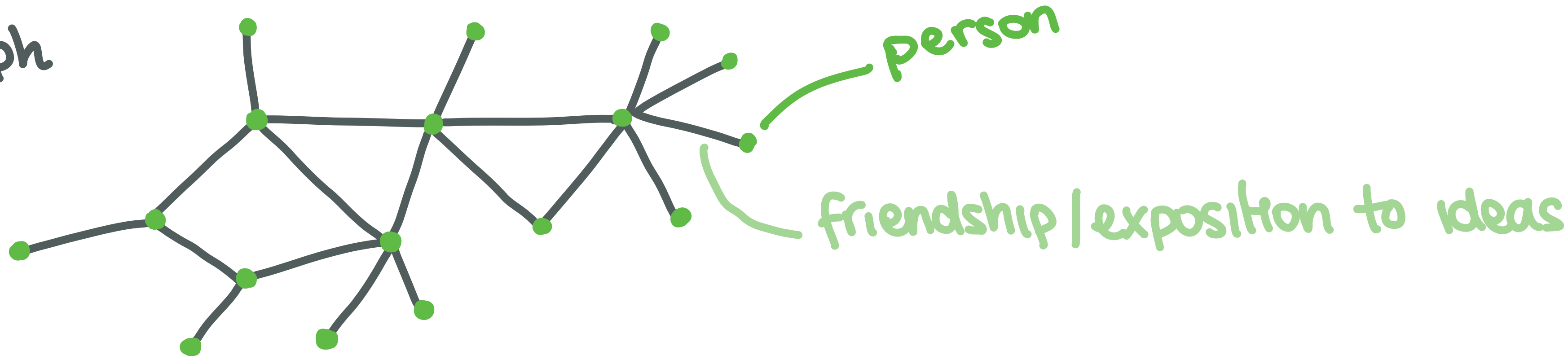
Opinion Dynamics on sheaves

Jakob Hansen, Robert Ghrist

K is a graph



$C^0(K;\mathcal{F})$	0-chains	private opinion distribution
$C^1(K;\mathcal{F})$	1-chains	pairwise discussion
$\partial^0: C^0 \longrightarrow C^1$	coboundary	aggregate of public disagreement
$L(C^0(K;\mathcal{F})$	laplacian	"average" of private disagreement
$H^0(K;\mathcal{F})$	global sections	harmonic opinions

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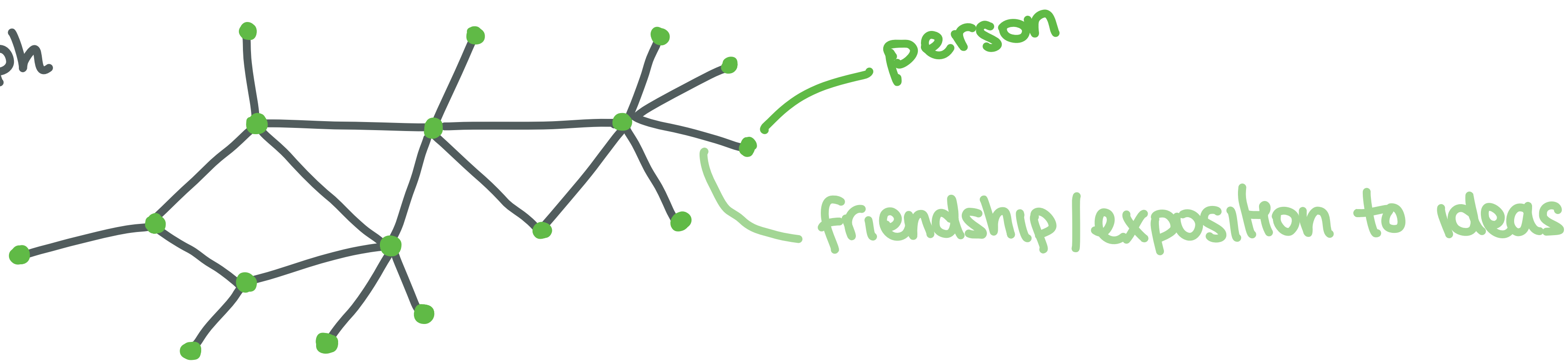

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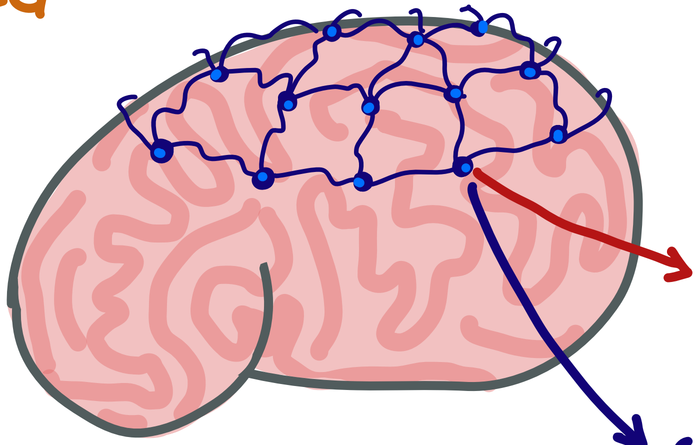
		public agreement

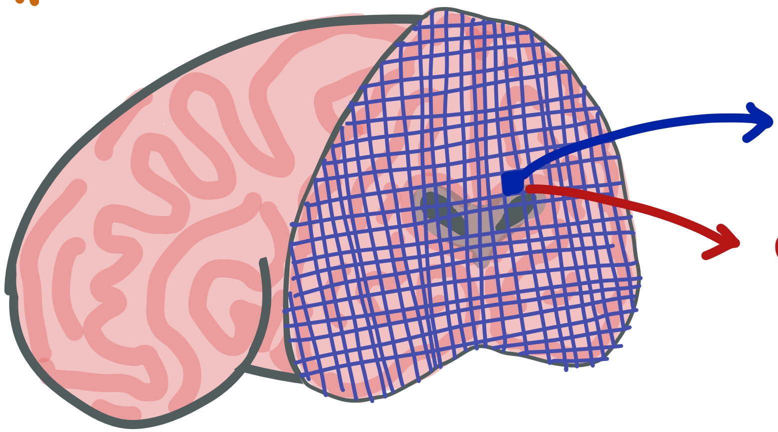
My research

TOPOLOGY + BRAIN FUNCTION

EEG



MRI



Topological Data Analysis: getting started

ARTICLES

Persistent Homology - Theory & Practice H. Edelsbrunner and D. Morozov
Barcodes: the persistent topology of data R. Ghrist
Topology and data G. Carlsson
High-dimensional Topological Data Analysis F. Chazal
Persistence theory: from quiver representations to data analysis S. Oudot

BOOKS

Elementary Applied Topology R. Ghrist
Computational Topology: an introduction H. Edelsbrunner and J.L. Harer
Topology for computing A.J. Zomorodian
Topological Signal Processing M. Robinson

Topological Data Analysis: the community

Applied algebraic topology network Youtube channel + weekly seminar series
WinCompTop - Women in computational topology Google group + newsletter

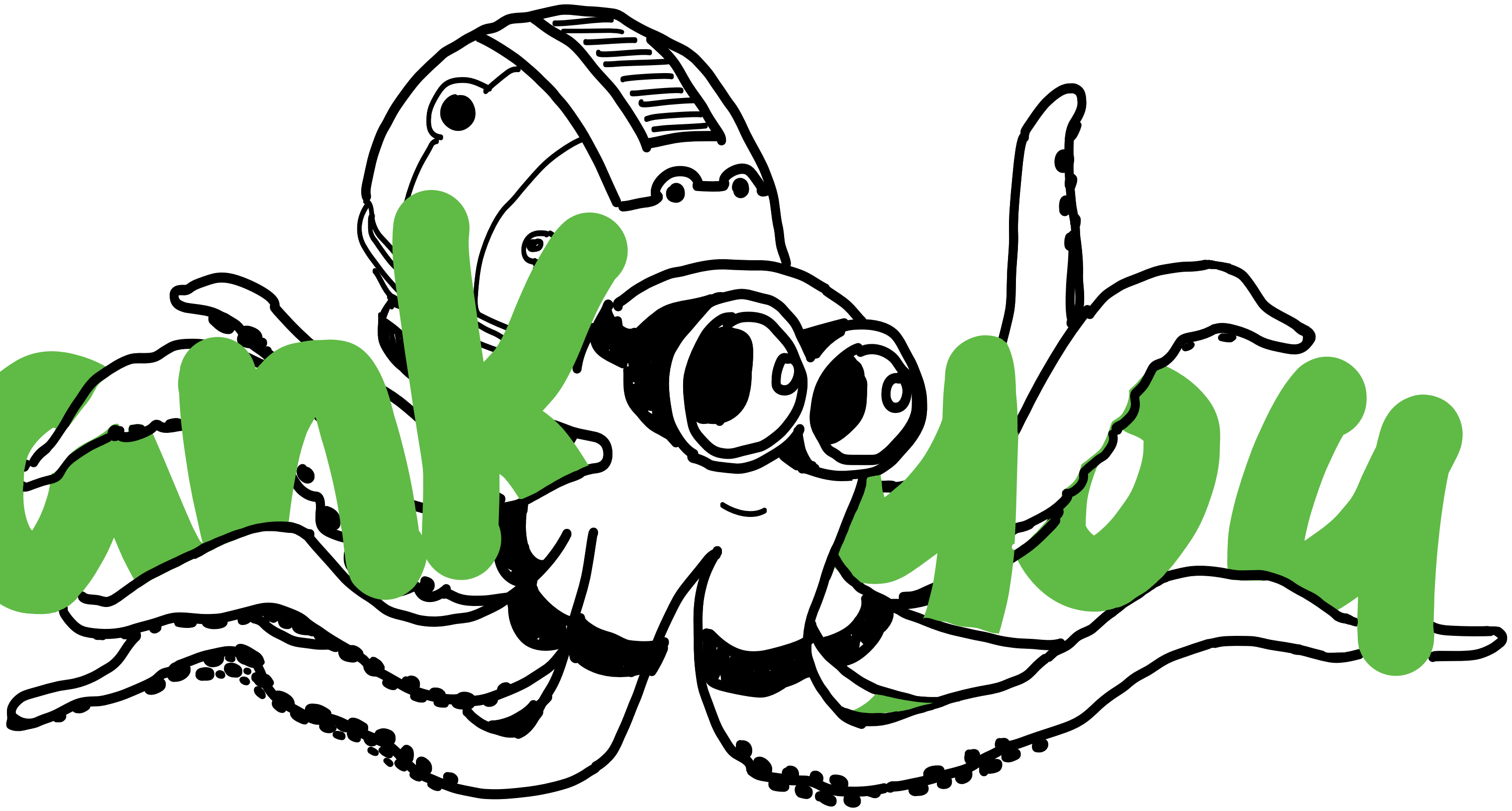
The conferences

ATMCS = Algebraic Topology: Methods, Computations and Science every 2 years
SOCG = Symposium on Computational Geometry every year

The journals

Journal of Applied and computational topology
Homotopy, Homology and Applications
SIAM Journal of Applied Algebra and Geometry
Discrete and Computational Geometry
Foundations of computational Mathematics

Thank you



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