HOMOLOGICAL METHODS FOR TEMPORAL NETWORKS

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Introduction

Our aim is to detect remote synchronization in arbitrary networks of coupled oscillators. We do so by producing appropriate delay embeddings of the network structure.

Method

- Frustrated Kuramoto model: oscillators are the nodes of a complex network, interactions include a phase frustration $\alpha > 0$.
- The systems reaches remote synchronization where the configuration of phases reflects the symmetries of the underlying coupling network, as proved in [3].
- A sequence of network snapshots of length d is mapped into a high-dimensional metric space.
- Extending previous works [1, 2], we are able to detect the remote synchronization regime through persistent homology.

Remote Synchronization

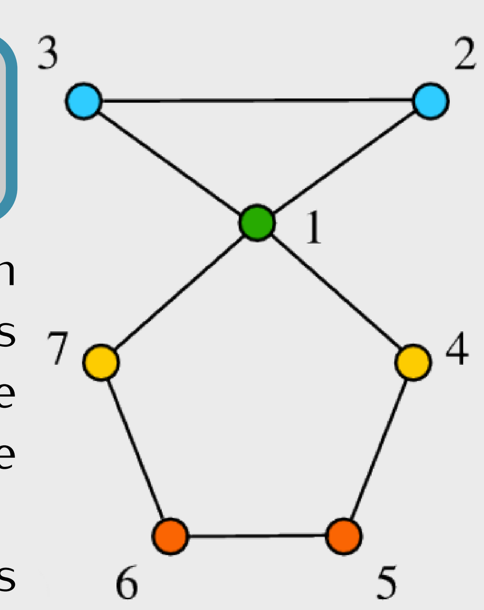
Frustrated Kuramoto Model

$$\frac{d\theta_i}{dt} = \omega_i + \frac{\lambda}{N}\sum_{j=1}^{N} sin(\theta_j - \theta_i - \alpha)$$

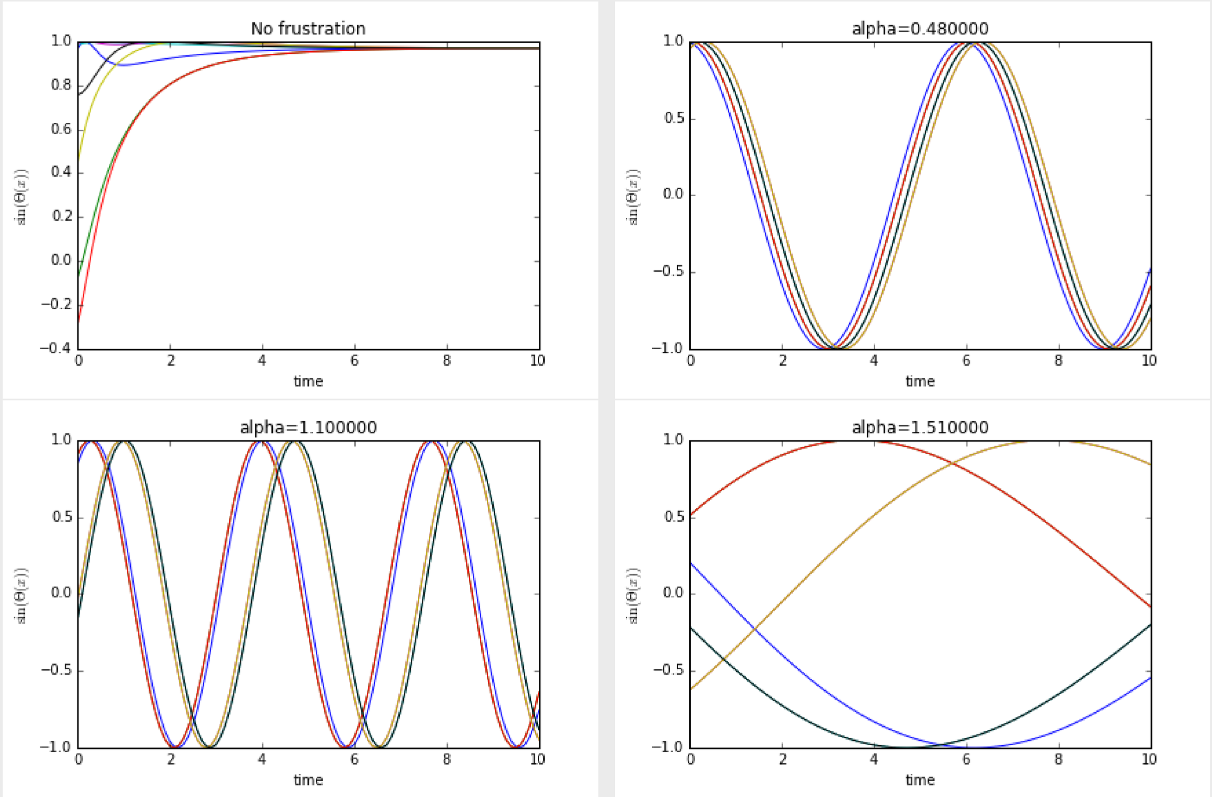
Each node of the complex network is an oscillator which continuously re-adjusts its frequency in order to minimize the difference between its phase and the phase of all the other oscillators.

The phase frustration parameter α forces connected nodes to maintain a finite phase difference.

After a transient period, the system reaches a phase-locked synchronized state in which symmetric nodes have the same phase.



The color code represents the phases of nodes at a given time in the stationary state.



When the system settles into a stationary state the phases are grouped into four different trajectories: $\theta_1(t)$, $\theta_2(t) = \theta_3(t)$, $\theta_4(t) = \theta_7(t)$ and $\theta_5(t) = \theta_6(t)$. By increasing the frustration parameter we better separate the four trajectories. The panels correspond to four different values of α.

Conclusions

Our method can detect the different trajectories of the phases of symmetric nodes through the persistent diagram.

Outlook: Study the circular coordinates of the cycles detected with non-costant frequencies.

Applications to real-world networks.

Time-delay embedding

Definition

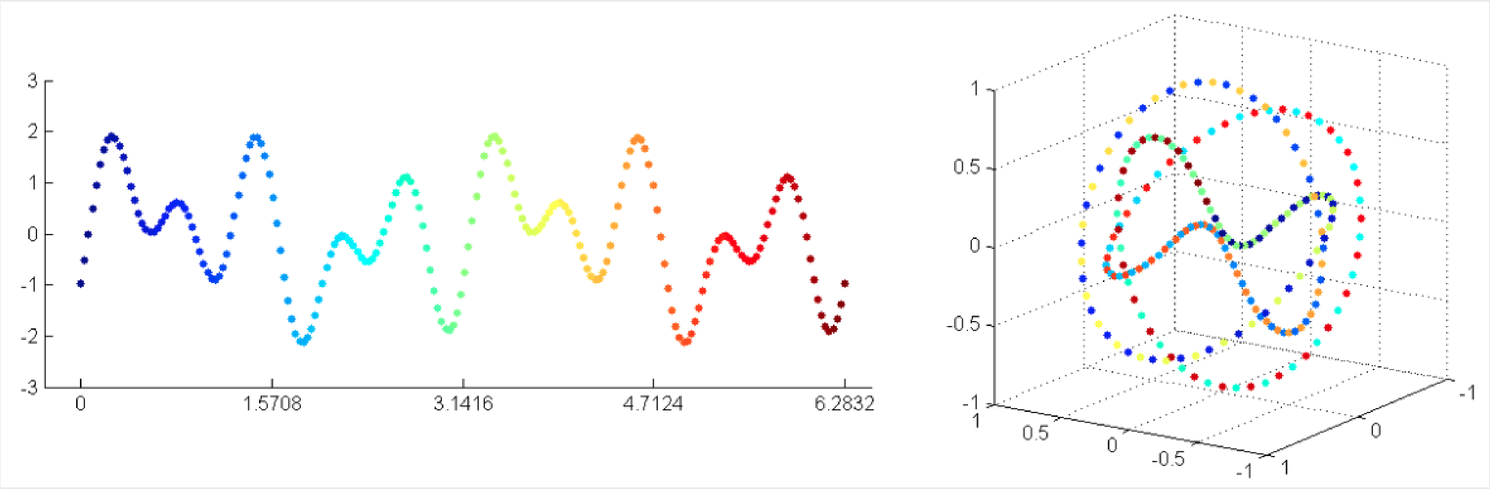
Given a time-series $f : t \to \mathbb{R}$, a **time-delay embedding** is a lift to a time-series $\phi : t \to \mathbb{R}^d$ defined by

$$\phi(t) = (f(t), f(t+\tau), \ldots, f(t+(d-1)\tau))$$

Takens theorem gives conditions under which a smooth attractor can be reconstructed from the observations of a function.

Takens Embedding Theorem (1981)

A smooth attractor can be reconstructed from the observations formed from time delayed values of the scalar measurements.

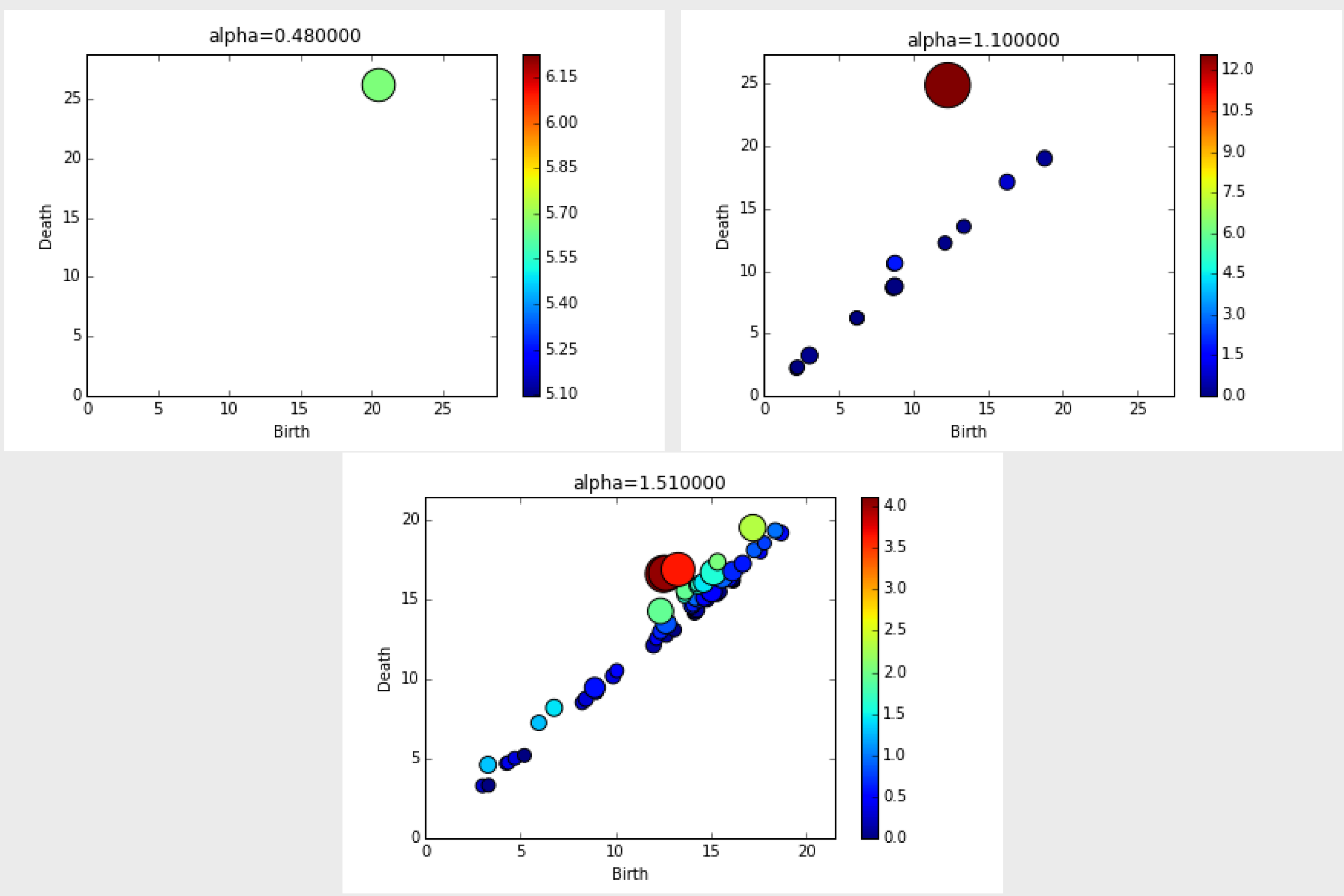


From a periodic time series to its time-delay embedding point cloud. Figure from Jose A. Perea, and John Harer (2012) [3]

Results

Periodicity and recurrence in dynamical systems are expressed as circles in the phase space.

We construct a simplicial complex approximating the topology of the embedded point cloud (eg. Vietoris Rips), and then create a persistence diagram for 1-dimensional homology.



Persistent diagrams of the system of oscillator at four different values of frustration parameter α. Each circle is associated to a point in the persistence diagram whose position indicates its robustness.

References

[1] De Silva, Skraba, and Vejdemo-Johansson. Topological Analysis of Recurrent Systems. Discr. & Comp. Geometry 45.4 (2011): 737-759.

[2] Perea, and Harer. Sliding windows and persistence: An application of topological methods to signal analysis. In NIPS Workshop on Algebraic Topology and Machine Learning (2012)

[3] Nicosia, et al. Remote synchronization reveals network symmetries and functional modules. PRL 110.17 (2013): 174102.